SOME GENERALIZATIONS OF GEODESICS*

BY

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1. Introduction

Let $y^{(k)}$ $(k = 1, 2, 3, 4)$ be four linearly independent analytic functions of two independent variables, u and v, and interpret $y^{(1)}, \ldots, y^{(4)}$ as the homogeneous coördinates of a point P_y in ordinary space. If the locus of P_y is a proper surface S, and if the curves u = const. and v = const. are the asymptotic lines of S, assumed to be distinct, $y^{(1)}, \ldots, y^{(4)}$ will form a fundamental system of solutions of a system of linear homogeneous differential equations of the form

$$\begin{aligned} &y_{uu} + 2ay_u + 2by_v + cy = 0, \\ &y_{vv} + 2a'y_u + 2b'y_v + c'y = 0, \end{aligned} \tag{1}$$

where $a, b, \ldots, c'$ are analytic functions of u and v, which must satisfy certain integrability conditions, one of which is

$$a_v = b'_u. \tag{1a}$$

If the coördinates $y^{(1)}, \ldots, y^{(4)}$ are homogeneous cartesian coördinates:

$$y^{(1)} = x, \quad y^{(2)} = y, \quad y^{(3)} = z, \quad y^{(4)} = 1, \tag{2}$$

we shall have

$$\begin{aligned} 2a &= -\left\{ {11 \atop 1} \right\}, \quad 2b = -\left\{ {11 \atop 2} \right\}, \quad c = 0, \\ 2a' &= -\left\{ {22 \atop 1} \right\}, \quad 2b' = -\left\{ {22 \atop 2} \right\}, \quad c' = 0, \end{aligned} \tag{2a}$$

where the Christoffel symbols are formed from the quadratic form

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2 \tag{3}$$

for the square of the arc element of S.

We propose to study the properties of certain two-parameter families of curves on S, namely those which are defined by a differential equation of the second order of the form

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$$v'' = A(v')^3 + 3B(v')^2 + 3Cv' + D, \quad v' = \frac{dv}{du}, \quad v'' = \frac{d^2v}{du^2}, \tag{4}$$

where A, B, C, D are analytic functions of u and v. In this paper, all such curves shall be called *hypergeodesics*. Moreover, all hypergeodesics which satisfy the same equation of form (4) shall be said to belong to the same *family*.

If we are using cartesian coördinates, these curves will reduce to *geodesics*, i. e., to extremal curves of the integral

$$s = \int \sqrt{Edu^2 + 2Fdudv + Gdv^2},$$

if the coefficients $A, \ldots, D$, are expressible in the form

$$A = \left\{ {22 \atop 1} \right\}, 3B = 2\left\{ {12 \atop 1} \right\} - \left\{ {22 \atop 2} \right\}, 3C = -2\left\{ {12 \atop 2} \right\} + \left\{ {11 \atop 1} \right\}, D = -\left\{ {11 \atop 2} \right\}. \tag{5}$$

If we consider any other integral of the form

$$s' = \int \sqrt{E'du^2 + 2F'dudv + G'dv^2}, \tag{6}$$

its extremal curves will also be hypergeodesics, satisfying an equation of form (4) with the coefficients

$$A' = \left\{ {22 \atop 1} \right\}', 3B' = 2\left\{ {12 \atop 1} \right\}' - \left\{ {22 \atop 2} \right\}', 3C' = -2\left\{ {12 \atop 2} \right\}' + \left\{ {11 \atop 1} \right\}', D' = -\left\{ {11 \atop 2} \right\}', \tag{7}$$

the new Christoffel symbols $\left\{ {ij \atop k} \right\}'$ being formed with respect to the quadratic form

$$ds'^2 = E'du^2 + 2F'dudv + G'dv^2. \tag{7a}$$

Since (4) contains four arbitrary functions of u and v as coefficients, while the coefficients (7) depend on only three arbitrary functions, E', F', G', those equations of form (4), whose coefficients are expressible in the form (7), define only a special class of hypergeodesics, which we shall call *quadratic extremal curves*. If we change from the surface S to a new surface S' whose squared arc element is given by (7a), these curves will become geodesics on S'.

Through a given point P of a surface pass ∞^1 geodesics, one for every tangent of P. Their osculating planes form a pencil with the surface normal of P as axis. For this reason we shall say that the congruence of normals is axially related to the surface. Let us consider *any congruence axially related to the surface*, so that one line l' of the congruence will pass through every point P of the surface without, however, being tangent to the surface at P. Such a congruence will determine a two-parameter family of curves on S, such that the osculating planes of the ∞^1 curves of the family which pass through P will form a pencil

with l' as axis. Such curves shall henceforth be called *axial union curves* of the surface with respect to the congruence Γ' formed by the lines l'.* Their differential equation is also of form (4), the coefficients being subject to the conditions

$$(8)\qquad A = -2a' = \left\{ {22 \atop 1} \right\},\ D = 2b = -\left\{ {11 \atop 2} \right\}.$$

Of course, a geodesic, being a quadratic extremal curve, as well as an axial union curve, satisfies both conditions (7) and (8).

Dualistic considerations lead to the notion of congruences *radially* related to a surface and *radial* union curves. *A congruence* Γ *is radially related to a surface* if one of its lines l is assigned to every tangent plane π of the surface without however being tangent to the surface at P. Instead of thinking of a curve C on S as a point locus, let us think of the one-parameter family of planes tangent to S along C, or their developable D, whose generators will be the tangents t' of S conjugate to the tangents t of C.

To the osculating plane of a point P of C will correspond, in this dualistic view, the point in which the corresponding conjugate tangent t' of P meets the cuspidal edge of D. This point has been used extensively so far only in the theory of conjugate nets and is there called a Laplace transform of P. In that theory, however, there arises a second point of this kind which is on t, and the line joining these two points is called the *ray of* P. For this reason we shall henceforth call the point in which the tangent t', conjugate to the tangent t of C at P, meets the cuspidal edge of D, the *ray point of* P with respect to the curve C. By analogy the osculating plane of C at P will sometimes be called the *axis plane* of P, for the line of intersection of the osculating planes for the curves C and C' of a conjugate net is called the *axis* of P.

To axial union curves will correspond, in this dualistic view, *radial union curves whose defining property is as follows; the ray point of each of its points lies on the corresponding line* l *of a radially related congruence.* Equation (4) will define radial union curves on S if and only if

$$(9)\qquad A = 2a' = -\left\{ {22 \atop 1} \right\},\ D = -2b = \left\{ {11 \atop 2} \right\}\dagger.$$

The curves of a pencil of conjugate nets‡ form a two-parameter family given by

* In Miss Sperry's thesis they are called *union curves*. Pauline Sperry, *Properties of a certain projectively defined two-parameter family of curves on a general surface*, American Journal of Mathematics, vol. 40 (1918), pp. 213–224. See also G. M. Green, *Memoir on the general theory of surfaces and rectilinear congruences.*, these Transactions, vol. 20 (1919) pp. 79–153.

† In Miss Sperry's thesis and in Green's memoir, quoted above, these curves are called *adjoint* or *dual* union curves. To obtain conditions (9) compare (4) and equation (132) of Green's memoir.

‡ E. J. Wilczynski, *Geometrical significance of isothermal conjugacy of a net of curves*, American Journal of Mathematics, vol. 42 (1920), p. 217.

$$\left(\frac{dv}{du}\right)^2 = k\mu,$$

where μ may be any function of u and v, and where k is an arbitrary constant. The second order differential equation of the curves of this family is

$$v'' = \frac{1}{2}\frac{\mu_v}{\mu}v'^2 + \frac{1}{2}\frac{\mu_u}{\mu}v',$$

which is again a special case of (4) characterized by the conditions

$$A = D = 0, \quad B_u = C_v. \tag{10}$$

We have given a number of instances in which differential equations of form (4) arise. They clearly form an important class. This paper is devoted to a discussion of their most striking geometric properties.

2. Properties of the osculating planes

Let C be an integral curve of equation (4) on a surface S defined by (1). The coördinates $y^{(k)}$, of a point P of C, may then be regarded as functions of u, and we find

$$\begin{aligned} \frac{dy}{du} &= y' = y_u + y_v v', \\ \frac{d^2y}{du^2} &= y'' = y_{uu} + 2y_{uv}v' + y_{vv}v'^2 + y_v v'', \end{aligned} \tag{11}$$

or, making use of (1),

$$y'' = -(c + c'v'^2)y - 2(a + a'v'^2)y_u + [-2(b + b'v'^2) + v'']y_v + 2v'y_{uv}. \tag{12}$$

If we substitute into (11) and (12) in succession the four solutions $y^{(k)}$ of (1), we obtain four points y, y', y'' whose plane is the osculating plane of C at P. Let us use the four points defined by y, y_u, y_v, y_{uv} as elements of a local coördinate system in the sense that the local coördinates of a point shall be proportional to x_1, x_2, x_3, x_4 if that point is defined by an expression of the form

$$x_1 y + x_2 y_u + x_3 y_v + x_4 y_{uv}.$$

Then *the equation of the plane which osculates the curve C at P will be*

$$2v'^2 x_2 - 2v'x_3 + [v'' + 2(a'v'^3 - b'v'^2 + av' - b)]x_4 = 0, \tag{13}$$

referred to the local tetrahedron of y, y_u, y_v, y_{uv}. So far we have made no use of (4). For an integral curve of (4), the coördinates of the osculating plane are

$$(14)\quad u_1=0, \qquad u_2=2v'^2, \qquad u_3=-2v',$$
$$u_4=(A+2a')v'^3+(3B-2b')v'^2+(3C+2a)v'+D-2b,$$

if the curve passes through P in the direction determined by v'. Since v' is arbitrary we obtain a one-parameter family of such planes, and (14) shows that their envelope is, in general, a cone of class three, called the *axis-plane cone* or *osc-cone* of P. The class reduces to two if either $D-2b$ or $A+2a'$ is equal to zero (but not both). The class reduces to one, so that the cone becomes a straight line, if and only if

$$D-2b=A+2a'=0$$

in accordance with the conditions (8), already noted, for an axial union curve.

If we eliminate v' from (14) we find the equation of the axis-plane cone in plane coördinates, viz.:

$$(15)\quad 2u_2u_3u_4+(A+2a')u_2^3-(3B-2b')u_2^2u_3$$
$$+(3C+2a)u_2u_3^2-(D-2b)u_3^3=0, \qquad u_1=0.$$

The cone has a double tangent plane, namely the plane $u_4=0$, which is tangent to S at P. This plane touches the cone along the elements $u_2=u_4=0$ and $u_3=u_4=0$, which are the asymptotic tangents of S at P. Consequently the cone is a surface of the *fourth order* and must possess three cuspidal tangent planes. The Hessian of (15) is equivalent to

$$(16)\quad 2u_2u_3u_4-3(A+2a')u_2^3-(3B-2b)u_2^2u_3+(3C+2a)u_2u_3^2+3(D-2b)u_3^3=0,$$

so that the coördinates of the three cuspidal tangent planes must satisfy

$$(17)\qquad (A+2a')u_2^3-(D-2b)u_3^3=0$$

as well as (15). But since (15) may be written

$$(18)\quad u_2u_3[2u_4-(3B-2b')u_2+(3C+2a)u_3]+(A+2a')u_2^3-(D-2b)u_3^3=0,$$

the coördinates of the three cuspidal tangent planes are given by

$$(19)\qquad \begin{aligned} u_1=0, \qquad (A+2a')u_2^3-(D-2b)u_3^3=0,\\ u_4+(b'-\tfrac{3}{2}B)u_2+(a+\tfrac{3}{2}C)u_3=0.\end{aligned}$$

Consequently these three planes intersect in the line

$$(20)\qquad u_1=0,\ u_4+(b'-\tfrac{3}{2}B)u_2+(a+\tfrac{3}{2}C)u_3=0,$$

which shall be called the *cusp-axis* of P with respect to the given two-parameter family of curves. The cusp-axis may be determined analytically as the line which joins the points y and

$$y_{uv}-(\tfrac{3}{2}B-b')y_u+(\tfrac{3}{2}C+a)y_v. \tag{21}$$

The cusp-axes of all of the points of S form a congruence axially related to S, called the *cusp-axis congruence* of the given two-parameter family of curves. The developables of this congruence correspond to a net of curves on S called the cusp-axis curves, which may be found most conveniently by making use of some general formulas developed by Green.* It should be noted, however, that Green's formulas do not apply directly to a general system of form (1) but only to the canonical form of such a system. If we put

$$y=l(u,v)\bar{y}, \tag{22a}$$

where

$$l_u=-al,\ l_v=-b'l, \tag{22b}$$

conditions which are consistent on account of (1a), system (1) is transformed into its canonical form

$$\bar{y}_{uu}+2b\bar{y}_v+f\bar{y}=0,\quad \bar{y}_{vv}+2a'\bar{y}_u+g\bar{y}=0, \tag{22c}$$

where

$$f=c-a_u-a^2-2bb',\quad g=c'-b'_v-b'^2-2aa'. \tag{22d}$$

This transformation may be applied to our equations and produces a result equivalent to putting a and b' equal to zero, and replacing c and c' by f and g, respectively. We find, in this way, *the differential equation of the cusp-axis curves*,

$$\begin{aligned}(f+2b_v-3bB+\tfrac{9}{4}C^2-\tfrac{3}{2}C_u)du^2-\tfrac{3}{2}(B_u+C_v)du\,dv&\\ -(g+2a'_u+3a'C+\tfrac{9}{4}B^2+\tfrac{3}{2}B_v)dv^2&=0.\end{aligned} \tag{23}$$

They form a conjugate net, if and only if

$$B_u+C_v\ =\ 0.$$

The three cuspidal tangent planes of the axis-plane cone intersect the tangent plane π, of P, in the three tangents

$$u_1=0,\ (A+2a')u_2^3-(D-2b)u_3^3=0.$$

If $A\ =\ D\ =\ 0$, as is the case if the two-parameter family consists of the curves of a pencil of conjugate nets, these three tangents coincide with the Segre tangents of the point P.† The same thing takes place, more generally, whenever

$$A=\omega a',\quad D=-\omega b, \tag{24}$$

where ω may be an arbitrary function of u and v.

* These Transactions, vol. 20 (1919), p. 90.

† We shall give a new definition for the Segre and Darboux tangents in §4.

Most of these results are due to Fubini.* We sum up the most important ones in the following theorem.

Consider any family of hypergeodesics on a surface S. Those which pass through a given point P, of S, form a one-parameter family. Their osculating planes, at P, envelop a quartic cone of class three, called the axis-plane cone or the osc-cone of P, which touches the tangent plane π along its asymptotic tangents, so that π is a double tangent plane of the cone. The osc-cone possesses three cuspidal tangent-planes which intersect along a line called the cusp-axis of P. Since the four functions, A, B, C, D, of u and v, may be chosen at pleasure, any system of quartic cones, associated with the given surface in this manner, may be regarded as corresponding to some equation of form (4). *Consequently the properties mentioned in this theorem are characteristic of a family of hypergeodesics.*

To every value of v' there corresponds a tangent t of S at P. The tangent t', which corresponds to $-v'$, is conjugate to t. The osculating plane of the corresponding hypergeodesic of the family (4) will be obtained from (14) by changing v' into $-v'$, giving

$$v_1=0, \quad v_2=2(v')^2, \quad v_3=2v',$$
$$v_4=-(A+2a')(v')^3+(3B-2b')(v')^2-(3C+2a)v'+D-2b$$

as the coördinates of this plane. If $x_1, \ldots, x_4$ are the coördinates of a point on the intersection of these two planes, we find

$$\rho x_2=-(3B-2b')(v')^2+D-2b,$$
$$\rho x_3=(A+2a')(v')^4+(3C+2a)(v')^2, \qquad \rho x_4=2(v')^2,$$

while x_1 remains arbitrary. Elimination of v' from these equations gives

$$[2x_2+(3B-2b')x_4][2x_3-(3C+2a)x_4]+(A+2a')(D-2b)x_4^2=0, \tag{25}$$

whence follows the following theorem.†

Let the two hypergeodesics of a family, which touch two conjugate tangents of a surface point P, be called conjugate hypergeodesics, and let the line of intersection of their osculating planes be called their axis. Then, the axes of all such pairs of conjugate hypergeodesics, for the same point P, form a quadric cone (25), *called the axis cone, which intersects the tangent plane π of P, in its asymptotic tangents, in such a way that the planes tangent to the cone along the asymptotic tangents intersect in the cusp-axis,* (20), *of P.*

* G. Fubini, *Fondamenti della geometria proiettivo-differenziale di una superficie*, Atti della Reale Accademia delle Scienze di Torino, vol. 53 (1918), p. 1037.

† This is a generalization of a corresponding theorem of Lane's concerned with a pencil of conjugate nets.

This quadric cone bears a very special relation to the surface S at P. The same thing is true of the quadric cone to which the osc-cone at P reduces when $(D - 2b)(A + 2a')$ is equal to zero. We may, instead, associate, with every point P of S, an arbitrary guadric cone

$$u_1=0, \quad \sum C_{ik}u_iu_k=0 \qquad (i, k=2, 3, 4),$$

having the point P as vertex, thus defining a certain complex whose lines are the elements of these ∞^2 cones. We may then define a two-parameter family of curves on S, *union curves* with respect to this complex, by demanding that the osculating plane ω of any point P of such a curve shall touch the corresponding cone. The differential equation of these curves will be

$$\begin{aligned} C_{44}[v'' + 2a(v')^3 - 2b'(v')^2 + 2a'v' - 2b]^2 \\ + 4v'(C_{42}v' - C_{34})[v'' + 2a(v')^3 - 2b(v')^2 + 2a'v' - 2b] \\ + 4C_{22}(v')^4 - 8C_{23}(v')^3 + 4C_{33}(v')^2 = 0. \end{aligned}$$

This equation is *quadratic* in v'', and reduces to the form (4) only if $C_{42} = C_{44} = 0$.

3. Application of Segre's correspondence

Toward the end of §1 we discussed very briefly the dualistic correspondence between the osculating plane ω of a curve C at P, and the corresponding ray-point of P. This correspondence was first studied in detail by Segre* and we shall hereafter speak of it as *Segre's correspondence.* To find the equations of this correspondence we observe that the homogeneous coördinates of the ray-point R of P, with respect to a curve C passing through P, are given by†

$$(26) \qquad \rho x_1 = v'' + 2(-a'v'^3 - b'v'^2 + av' + b), \quad \rho x_2 = 2v', \quad \rho x_3 = -2v'^2, \quad \rho x_4 = 0.$$

On the other hand the coördinates of ω, the osculating plane of C at P, are given by (13). The equations of Segre's correspondence are obtained from (13) and (26) by eliminating v' and v''. They are

$$(27) \qquad \rho x_1 = u_2u_3u_4 + 2a'u_2^3 + 2bu_3^3, \quad \rho x_2 = -u_2u_3^2, \quad \rho x_3 = -u_2^2u_3, \quad \rho x_4 = 0,$$

and

$$(28) \qquad \sigma u_1 = 0, \quad \sigma u_2 = -x_2x_3^2, \quad \sigma u_3 = -x_2^2x_3, \quad \sigma u_4 = x_1x_2x_3 + 2bx_2^3 + 2a'x_3^3,$$

where ρ and σ are factors of proportionality.

* Segre, *Complementi alla teoria delle tangenti coniugate di una superficie*, Rendiconti della R. Accademia dei Lincei, Series 5, vol. 17 (1908), pp. 405–412.

† Slightly generalized from the formulas given in Green's memoir (these Transactions, vol. 20 (1919), p. 131).

This *cubic birational correspondence* is very convenient for the purpose of developing the formulas which correspond to those of §2 by duality. To the quartic cone of class three corresponds a cubic curve of class four in the tangent plane, namely

$$(29)\qquad \begin{aligned} &x_4=0,\\ &2x_1x_2x_3+(D+2b)x_2^3-(3C+2a)x_2^2x_3+(3B-2b')x_2x_3^2-(A-2a')x_3^3=0, \end{aligned}$$

as the locus of the ray-points of those curves, of the family defined by (4), *which pass through P. This cubic curve has three points of inflection which are on the line*

$$(30)\qquad x_4=0,\ 2x_1-(3C+2a)x_2+(3B-2b')x_3=0,$$

which is called the flex-ray of P. Since the expressions

$$(31)\qquad y_u+(\tfrac{3}{2}C+a)y,\qquad y_v-(\tfrac{3}{2}B-b')y$$

represent the points of intersection of the asymptotic tangents of P with this flex-ray, we see by comparing with (21), that *the cusp-axis and the flex-ray of P with respect to the two-parameter family of curves defined by* (4) *are Green reciprocals of each other.**

If $D+2b=A-2a'=0$, the cubic curve (29) reduces to three straight lines, the asymptotic tangents of P, and the corresponding flex-ray. This corresponds to the case when equation (4) defines a two-parameter family of radial union curves.

The congruence formed by the flex-rays of all of the points of S is radially related to S. The curves on S which correspond to its developables, *the flex-ray curves of S, are determined by the differential equation*

$$(32)\qquad \begin{aligned} &(f-3bB+\tfrac{9}{4}C^2-\tfrac{3}{2}C_u)du^2-\tfrac{3}{2}(B_u+C_v)dudv\\ &\qquad\qquad -(g-3a'C+\tfrac{9}{4}B^2+\tfrac{3}{2}B_v)dv^2=0. \end{aligned}$$

The three points of inflection of the ray-point cubic satisfy equations (30) and

$$(D+2b)x_2^3-(A-2a')x_3^3=0.$$

When the two-parameter family of curves defined by (4) consists of the curves of a pencil of conjugate nets, $A=D=0$, and the above equation shows that the three points of inflection are on the three Darboux tangents of P. The same thing will be true more generally under the conditions (24).

Thus we see that *the cuspidal tangent planes of the axis-plane cone will intersect the tangent plane of P in its Segre tangents if the three points of inflection of the ray-point cubic are on the Darboux tangents and conversely*, except in the cases

* This relation will be discussed more in detail in §4.

in which the axis-plane cone or the ray-point cubic reduce to loci of the second or lower degree.

The ray-points of two conjugate tangents determine, as their line of junction, *the ray of the point P* with respect to a given pair of conjugate hypergeodesics. The coördinates of such a ray, corresponding to the pair $(v', -v')$ are

$$u_1 = -2v'^2, \quad u_2 = (A-2a')v'^4 + (3C+2a)v'^2, \quad u_3 = -(3B-2b')v'^2 - (D+2b),$$

while u_4 remains arbitrary. By eliminating v', we find

$$(33a) \qquad [2u_2 + (3C+2a)u_1][2u_3 - (3B-2b')u_1] + (A-2a)(D+2b)u_1^2 = 0,$$

the equation of the *ray conic* in plane coördinates, whence we may deduce

$$(33b) \qquad [2x_1 - (3C+2a)x_2 + (3B-2b')x_3]^2 + 4(A-2a')(D+2b)x_2x_3 = 0, \; x_4 = 0,$$

the equations of the ray conic in point coördinates. We have the following theorem.

The ray-points of a pair of conjugate hypergeodesics through a given point P determine the ray of the pair. The envelope of the rays of all such pairs, for a given point P, is a conic called the ray conic of P with respect to the given family of hypergeodesics. This conic is tangent to the asymptotic tangents of P, and the polar of P with respect to the conic coincides with the flex-ray of P. The ray conic touches the ray-point cubic in three points which are on the three tangents conjugate respectively to the three tangents which contain the points of inflection of the ray-point cubic.

4. Relation between Segre's and Green's correspondences

Two lines, l and l', are called Green reciprocals of each other with respect to the asymptotic net of a surface S, if l is a line not containing P of the tangent plane π of a point P of S, if l' passes through P but does not lie in its tangent plane, and if l and l' are reciprocal polars of each other with respect to the osculating quadric Q of P. If l is the line of junction of the two points

$$(34a) \qquad y_u - \beta y, \quad y_v - \alpha y,$$

l' will be the line which joins y to

$$(34b) \qquad z = y_{uv} - \alpha y_u - \beta y_v.$$

The equations

$$(35) \qquad x_2 + \alpha x_4 = 0, \quad x_3 + \beta x_4 = 0,$$

represent two planes through l', so that

$$\lambda x_2 + \mu x_3 + (\lambda\alpha + \mu\beta)x_4 = 0$$

will represent any plane through l'. This plane has the coördinates

$$(0, \lambda, \mu, \lambda\alpha + \mu\beta).$$

The points which correspond to the planes of the pencil whose axis is l', in Segre's correspondence, are given by

$$\rho x_1 = \lambda\mu(\alpha\lambda + \beta\mu) + 2a'\lambda^3 + 2b\mu^3, \quad \rho x_2 = -\lambda\mu^2, \quad \rho x_3 = -\lambda^2\mu, \quad \rho x_4 = 0.$$

If we eliminate λ and μ, we find that *the plane cubic curve*

$$(x_1 + \beta x_2 + \alpha x_3)x_2x_3 + 2(bx_2^3 + a'x_3^3) = 0, \quad x_4 = 0 \tag{36}$$

corresponds to the straight line (35) *in Segre's correspondence. It has a double point at P and the asymptotic tangents of P as double point tangents. Its three points of inflection are given by*

$$x_4 = 0, \quad bx_2^3 + a'x_3^3 = 0, \quad x_1 + \beta x_2 + \alpha x_3 = 0. \tag{37}$$

Consequently *these points of inflection are on the Darboux tangents of P*. We may regard this fact as a definition of the Darboux tangents. It is, in fact, simpler than any of the properties of the Darboux tangents which have so far been used for defining them. Since the three points of inflection are also on the line

$$x_1 + \beta x_2 + \alpha x_3 = 0,$$

which is the Green reciprocal of l', we obtain the following additional result.

The Green reciprocal of a line l' through P coincides with the flex-ray of the cubic curve which corresponds to l' in Segre's correspondence.

In similar fashion we find that *in the Segre correspondence, the cone*

$$u_1 = 0, \quad (u_4 - \alpha u_2 - \beta u_3)u_2u_3 + 2(a'u_2^3 + bu_3^3) = 0$$

corresponds to the line l, whose equations are

$$x_1 + \beta x_2 + \alpha x_3 = 0, \quad x_4 = 0.$$

This cone touches the tangent plane of P along the asymptotic tangents. Its three cuspidal tangent planes are given by

$$u_1 = 0, \quad a'u_2^3 + bu_3^3 = 0, \qquad u_4 - \alpha u_2 - \beta u_3 = 0.$$

They intersect the tangent plane of P in its Segre tangents (*thus providing a new definition for the Segre tangents*) *and have in common, as cusp-axis, the line l'. Thus the Green reciprocal of l coincides with the cusp axis of the quartic cone which corresponds to l in Segre's correspondence.*

5. Fubini's integral invariant and Green's pseudo-normal

The integral

$$I = \int \sqrt{a'bdudv} = \int \sqrt{a'bv'}du \tag{38}$$

is invariant for all transformations of the form

$$\bar{u} = \alpha(u), \ \bar{v} = \beta(v), \ \bar{y} = \lambda(u,v)y.$$

Consequently, the value of this integral extended over an arc of a curve C on S represents a quantity intrinsically and projectively determined by this arc. If the asymptotic curves of S had not been chosen as parametric curves, the integrand of I would differ from

$$\sqrt{Ddu^2 + 2D'dudv + D''dv^2},$$

the square-root of the second fundamental form of S, only by a factor which makes the product a projective invariant of the arc. This factor moreover is the projective differential invariant whose vanishing is characteristic of ruled surfaces.

Fubini has proposed to introduce this integral I as a projectively defined substitute for the notion of length of arc. The extremal curves of I may then be regarded as a projectively defined substitute for the geodesics of the metric theory. These extremals are given by

$$v'' + \frac{(a'b)_v}{a'b} v'^2 - \frac{(a'b)_u}{a'b} v' = 0, \tag{39}$$

an equation of form (4) for which

$$A = 0, \ 3B = -\frac{(a'b)_v}{a'b}, \ 3C = \frac{(a'b)_u}{a'b}, \ D = 0. \tag{40}$$

The corresponding cusp-axis is the line which joins y to the point

$$y_{uv} + \left[b' - \tfrac{1}{2}\left(\frac{a'_v}{a'} + \frac{b_v}{b}\right)\right] y_u + \left[a - \tfrac{1}{2}\left(\frac{a'_u}{a'} + \frac{b_u}{b}\right)\right] y_v; \tag{41}$$

since $B_u + C_v = 0$, the corresponding cusp-axis curves and flex-ray curves form conjugate nets.

*This formula shows that the cusp-axis of a surface point with respect to the extremal curves of Fubini's integral invariant coincides with the line which Green has called the pseudo-normal, and which was defined by him in an entirely different fashion.**

* See Green's Memoir, pp. 125–127.

Green and Fubini were struck, quite independently, with the analogy which exists between this line and the normal, and both of them sketched in outline far-going theories based upon this analogy. From Green's point of view, the principal analogies are these: the pseudo-normal, like the normal, is intrinsically connected with the surface, and the curves which correspond to the developables of the pseudo-normal congruence resemble the lines of curvature, which correspond to the developables of the normal congruence, by forming also a *conjugate* net. Fubini's considerations allow us to add that both normal and pseudo-normal may be defined as cusp-axes of certain integral invariants. But if we follow Fubini in taking this latter property as a definition for the pseudo-normal, an essential gap remains to be filled. For, while Fubini has shown that the integral I is intrinsically and projectively related to an arc of a curve on S, he has not explained the nature of this relation. Green's definition, by way of contrast, although somewhat complicated, is perfectly complete. We propose to complete Fubini's definition also, by providing a geometric interpretation for the integral I.

6. Interpretation of Fubini's integral invariant

Let us select a family of hypergeodesics on the surface S, and arrange the curves of the family which pass through a given point P, of S, into pairs of conjugates, one pair of hypergeodesics for every pair of conjugate tangents. Each of these tangents contains the ray-point of P with respect to the hypergeodesic which belongs to the conjugate tangent, and we have called the line joining these ray-points the ray of P with respect to such a pair of conjugate hypergeodesics. The envelope of these rays, for all pairs of conjugate hypergeodesics P, we called the ray conic of P with respect to the selected family of hypergeodesics, and its equations we found to be

$$x_4=0,\ [2x_1-(3C+2a)x_2+(3B-2b')x_3]^2+4(A-2a')(D+2b)x_2x_3=0. \tag{33b}$$

Let C be a finite arc of a real continuous curve joining two points, A and B, of the surface S, and let (u_0, v_0) and (U, V) be the values of u and v which correspond to the points A and B. Let ϵ be a positive number, and divide the arc C by means of intermediate points $P_1, P_2, \ldots, P_{n-1}$, into n smaller arcs. Let the coördinates of P_k be (u_k, v_k), where $u_n=U$, $v_n=V$, let

$$\delta u_k=u_k-u_{k-1}, \quad \delta v_k=v_k-v_{k-1},$$

and let

$$\sqrt{\delta u_k^2+\delta v_k^2}\leqq\epsilon \qquad (k=1, 2, \ldots, n-1).$$

We associate, with every point P of C, the corresponding ray conic. These conics generate a surface Σ. As n grows beyond bound and ϵ approaches the

limit zero, the line $P_{k-1}P_k$ approaches a tangent of C as limit. Consequently there will be two points, R_{k-1} and R'_{k-1}, among the intersections of $P_{k-1}P_k$ with Σ, which will tend toward the points in which the tangent to C at P_{k-1} intersects the corresponding ray conic.

Now the tangent to C at P_{k-1} intersects the corresponding ray conic (33b) in two points whose local coördinates are proportional to

$$(42)\quad \begin{aligned} x_1 &= a_{k-1}+b'_{k-1}+\tfrac{3}{2}(C_{k-1}-B_{k-1})\pm 2\sqrt{(a'_{k-1}-\tfrac{1}{2}A_{k-1})(b_{k-1}+\tfrac{1}{2}D_{k-1})v'_{k-1}}\\ x_2 &= 1, \qquad x_3=v'_{k-1}, \qquad x_4=0, \end{aligned}$$

where we are thinking of v and $v' = dv/du$ as a function of u along the curve C, and where we have written

$$a_{k-1}=a(u_{k-1}, v_{k-1}), \text{ etc.}$$

The equations of $P_{k-1}P_k$ differ from those of the tangent to C at P_{k-1} only by terms of order ϵ^2. Therefore, the intersections, R_{k-1} and R'_{k-1}, of $P_{k-1}P_k$ with Σ will differ from the points (42), the intersections of the tangent to C at P_{k-1} with Σ, only by terms of order ϵ^2. In computing the double-ratio of the four points R_{k-1}, R'_{k-1}, P_{k-1}, P_k to terms of order ϵ, we may therefore use the values given by (42) for the coördinates of R_{k-1} and R'_{k-1}.

We find therefore

$$(43)\quad \begin{aligned} &(R_{k-1}, R'_{k-1}, P_{k-1}, P_k)\\ &\qquad =1+4\sqrt{(a'_{k-1}-\tfrac{1}{2}A_{k-1})(b_{k-1}+\tfrac{1}{2}D_{k-1})v'_{k-1}}\,\delta u_k+(\epsilon^2)_k, \end{aligned}$$

where $(\epsilon^2)_k$ represents a quantity of order ϵ^2.

The integrand of Fubini's integral will appear in this equation if we assume $A = D = 0$, although a somewhat more general hypothesis would accomplish the same result. We shall, in fact, make the still more special assumption

$$A=D=0, \quad B_u=C_v,$$

so that the hypergeodesics which we are using become curves of a pencil of conjugate nets. Then (43) reduces to

$$(43a)\qquad (R_{k-1}, R'_{k-1}, P_{k-1}, P_k)=1+4\sqrt{a'_{k-1}b_{k-1}v'_{k-1}}\,\delta u_k+(\epsilon^2)_k,$$

an equation which is fundamental for our purpose.

We have associated with every point P of the arc C a conic in the corresponding tangent plane π, namely the ray conic of P with respect to an associated pencil of conjugate nets. Let us use this conic as a basis for a definition, in Cayley's sense, of non-euclidean distances in the plane π. If M and N are two distinct points of π and if R and R' are the points of intersection of the line MN

with the ray conic of this plane, we define the non-euclidean distance of MN to be

$$\Delta_{MN} = \tfrac{1}{4} \log (R, R', M, N).* \tag{44}$$

From (43) we find

$$\Delta_{P_{k-1}P_k} = \sqrt{a'_{k-1}b_{k-1}v'_{k-1}}\,\delta u_k + (\epsilon^2)_k.$$

Thus *we may look upon the elements of Fubini's integral as infinitesimal non-euclidean distances between neighboring points of the given arc, each of these infinitesimal distances being measured with respect to a different conic as absolute. The integral itself is the limit of a sum of such infinitesimal non-euclidean distances.* It is not, in general, itself a non-euclidean distance in the classical sense because, in the classical definition of a non-euclidean distance, the same quadric locus serves as absolute for all points of space.

Another interpretation of the Fubini integral may be obtained as follows. From (43a) we find

$$\begin{aligned}(R_0, R'_0, A, P_1) &= 1 + 4\sqrt{a'_0b_0v'_0}\,\delta u_1 + (\epsilon^2)_1,\\ (R_1, R'_1, P_1, P_2) &= 1 + 4\sqrt{a'_1b_1v'_1}\,\delta u_2 + (\epsilon^2)_2.\end{aligned} \tag{45}$$

Let π_0 denote the perspectivity which transforms

$$R_0,\ R'_0,\ P_1 \text{ into } R_1,\ R'_1,\ P_1$$

respectively, and let A' be the point of P_1P_2 which corresponds to A in this correspondence. Then we may replace the first equation of (44) by

$$(R_1, R'_1, A', P_1) = 1 + 4\sqrt{a'_0b_0v'_0}\,\delta u_1 + (\epsilon^2)_1. \tag{46}$$

On account of the familiar double-ratio equation

$$(ABCD)\ (ABDE) = (ABCE),$$

we find from (45) and (46), by multiplication,

$$(R_1, R'_1, A', P_2) = [1 + 4\sqrt{a'_0b_0v'_0}\,\delta u_1 + (\epsilon^2)_1]\,[1 + 4\sqrt{a'_1b_1v'_1}\,\delta u_2 + (\epsilon^2)_2]. \tag{47}$$

The four points $R_1\ R'_1,\ A',\ P_2$ are on P_1P_2. We determine the perspectivity π_1 which projects

$$R_1, R'_1, P_2 \text{ into } R_2,\ R'_2,\ P_2$$

respectively, and denote by A'' the point of P_2P_3 which corresponds to A'.

* We may also define non-euclidean *angles* with respect to the axis cone.

We may then replace the left member of (47) by (R_2, R_2', A'', P_2). If we multiply both members of the resulting equation by those of

$$(R_2, R_2', P_2, P_3) = 1 + 4\sqrt{a_2'b_2v_2'}\delta u_3 + (\epsilon^2)_3,$$

we find

$$(48) \qquad (R_2, R_2', A'', P_3) = [1 + 4\sqrt{a_0'b_0v_0'}\delta u_1 + (\epsilon^2)_1]\,[1 + 4\sqrt{a_1'b_1v_0'}\delta u_2 + (\epsilon^2)_2]\,[1 + 4\sqrt{a_2'b_2v_2'}\delta u_3 + (\epsilon^2)_3].$$

Clearly we may continue in this way, obtaining the equation

$$(49) \qquad (R_{n-1}, R_{n-1}', A^{(n-1)}, B) = \prod_{k=1}^{n} [1 + 4\sqrt{a_{k-1}'b_{k-1}v_{k-1}'}\delta u_k + (\epsilon^2)_k],$$

where B is the end-point of the arc under consideration, and where $A^{(n-1)}$ is a point on $P_{n-1}B$ which is derived from A, the initial point of the arc, by the sequence of perspectivities here described. It is easy to see, by familiar methods, that the product in the right member of (49) will differ from

$$\prod_{k=1}^{n} [1 + 4\sqrt{a_{k-1}'b_{k-1}v_{k-1}'}\delta u_k]$$

by terms of order ϵ at most, and that this product will approach a limit when n grows beyond bound, provided that the definite integral

$$\int_{(u_0, v_0)}^{(U, V)} \sqrt{a'bv'}du$$

exists.

Let R and R' be the points in which the tangent to C at B intersects the corresponding ray conic, and let A^* be the point on this tangent which $A^{(n-1)}$ approaches when n grows beyond bound. We find

$$(50) \qquad \kappa = (R, R', A^*, B) = \lim_{n \to \infty} \prod_{k=1}^{n} [1 + 4\sqrt{a_{k-1}'b_k\delta u_k \delta v_k}],$$

and

$$(51) \qquad \int_A^B \sqrt{a'bdudv} = \tfrac{1}{4}\log\kappa,$$

thus defining Fubini's integral as one-fourth of the logarithm of a certain cross ratio which is defined purely projectively, by a process which constitutes a multiplicative analogon of a definite integral.

In this theorem it remains to prove (51). We observe that κ is defined by (50) as a function of U and V, the coördinates of the end-point B of the arc C.

Let us assume that the integral $\int \sqrt{a'bdudv}$ still exists when the arc C is extended by adding to it the arc BB' where the coördinates of B' are $U + dU$, $V + dV$. Then

$$\kappa(U + dU, V + dV) = \kappa(U, V)\,[1 + 4\sqrt{a'(U,V)b(U,V)dUdV}] + (\epsilon^2),$$

and therefore

$$\frac{\log \kappa(U + dU, U + dU) - \log \kappa(U, V)}{dU}$$

will differ from

$$4\sqrt{a'(U, V)b(U, V)\frac{dV}{dU}}$$

at most by an infinitesimal of order ϵ. Consequently we have

$$\frac{d \log \kappa(U, V)}{dU} = 4\sqrt{a'(U)b(V)\frac{dV}{dU}},$$

which is equivalent to (51).

The methods of this article make it possible to interpret many other integral invariants of a curve on a surface. We need merely replace the ray conic by some other conic or pair of lines.

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ON THE GYROSCOPE

BY

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1. The Problem and its Solution

The problem which led to the results of this paper was the following.* Consider a gyroscope† which is rotating with high velocity about its axis, the center of gravity, O, being fixed in space. Let a couple, $\mathfrak{M}$, be applied. The axis of the gyroscope will in general describe a cone, $\mathfrak{K}$, whose vertex is at O. *What is the relation between the geometric configuration* (i.e., the cone $\mathfrak{K}$), *the velocity with which the axis sweeps it out, and the couple?*

Instead of the center of gravity, any other point O of the axis may be chosen as the point of the body which is to be fixed in space. Let any forces be applied, and consider the vector moment of each with respect to O. The sum of these vectors shall be denoted by $\mathfrak{M}$, and the question in its new formulation is clear.

Again, let the gyroscope be free to move in space under the action of any forces. Its rotation is the same as that of an equivalent gyroscope whose center of gravity is held fast, and which is acted on by the same relative forces. The motion of the latter gyroscope is precisely the case discussed here.

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* Since returning the galleys my attention has been called to Lamb, *Higher Mechanics*, 1920, p. 133, where the formulas appear: $Av\,d\chi/dt = Q + Cnv$, $A\,dv/dt = P$. Mr. Lamb obtains a highly interesting result by interpreting these equations in terms of the motion of a particle of mass A constrained to slide on a smooth sphere and acted on by a force tangent to the sphere, whose components are P and Q, and in addition by a force along the normal to the path, equal to Cnv. He does not, however, make the cone $\mathfrak{K}$ of the present article an element of his discussion, and so of course does not introduce the conception of the bending of this cone; nor does he mention the geodesic curvature of the curve we have called $\mathfrak{C}$. Moreover, in his discussion of the rising of the axis of the top with blunt peg, spinning on a rough table, he arrives at the usual false result (p. 136) that friction always causes the axis to rise; cf. §6 of the present article. The present paper was sent to the Transactions on March 19, 1921, and presented to the Society at its April meeting, 1921; cf. Bulletin of the American Mathematical Society, vol. 27 (1921), p. 391.

† By a *gyroscope* we mean a rigid body having an axis of material symmetry. More precisely, there is a line in the body, called the axis, which has the following properties: (i) it passes through the center of gravity; (ii) if O is any point of this line, the latter coincides with one of the principal axes of inertia through O; denote the moment of inertia about this axis by C; (iii) the moments of inertia about all other principal axes through O are equal; denote their common value by A; (iv) A and C are both positive.

Let ζ be a point of intersection of the axis of the gyroscope with the unit sphere about O as center; and let ζ be so chosen that an observer at O, looking toward ζ, would see the gyroscope rotating in the positive sense, which we shall take as counter-clockwise. Let $\mathfrak{C}$ be the curve on the unit sphere traced out by ζ. It may be regarded as the directrix of the cone $\mathfrak{K}$.

We next introduce a measure of the *bending* of the cone $\mathfrak{K}$. Suppose that, fixing our attention on an arbitrary cone $\mathfrak{K}$ with its vertex at O, we allow ζ, the point in which a generating ray meets the unit sphere, to describe its path $\mathfrak{C}$ with unit velocity. Then the rate at which the tangent plane to $\mathfrak{K}$ at ζ is turning shall be defined as the *bending* of the cone, and denoted by κ.

In other words, let P be an arbitrary point of $\mathfrak{C}$, and P' a neighboring point. Let ϵ denote the angle between the tangent planes at P and P'. Then

$$\kappa = \lim_{P' \to P} \frac{\epsilon}{\overset{\frown}{PP'}}.$$

The angle ϵ is also the angle between the normals to the tangent planes. Denote the point of intersection of such a normal, thought of as directed and drawn from O, with the unit sphere by Q. Then κ is the rate at which Q is moving when P moves with unit velocity.

It is convenient, however, to introduce κ as an algebraic quantity. Let s be so chosen that, as ζ describes $\mathfrak{C}$, s increases. Then $v = ds/dt$ is positive. The tangent to $\mathfrak{C}$ in the sense of the increasing s shall be called the *positive tangent*, and the normal drawn to the right in the tangent plane of the sphere at P, when the observer, outside the sphere, is walking along $\mathfrak{C}$ in the positive sense, the *positive normal*;* cf. Fig. 1. And now κ shall be taken as positive when $\mathfrak{C}$ near P lies to the *left* of the great circle arc which is tangent to $\mathfrak{C}$ at P, i. e., on the side of the negative normal; and as negative, when the opposite is the case.

The concept of the bending, κ, is akin to that of the curvature of the cone $\mathfrak{K}$, thought of as the one-dimensional locus of its generators. But this idea of curvature has nothing to do with the curvature of $\mathfrak{K}$, when thought of as a two-dimensional manifold.

The bending, κ, of $\mathfrak{K}$ is identical (save, possibly, as to sign) with the geodesic curvature of $\mathfrak{C}$, regarded as a curve on the sphere. Moreover, κ is connected with the curvature, K, of $\mathfrak{C}$, regarded as a space curve, by the relation:

$$\kappa^2 = K^2 - 1.$$

Finally, κ is related to the torsion, T, of $\mathfrak{C}$ by the equation†

* More precisely, the positive normal at P is oriented with respect to the positive tangent and the direction of OP as the positive y-axis is oriented with respect to the positive axes of x and z.

† This latter relation was pointed out to me by Professor C. N. Haskins, to whom I am indebted for a careful examination of this paper, and for valuable suggestions.

$$\frac{d}{ds}\tan^{-1}\kappa = \pm T.$$

A case of great interest is the restricted one in which $\mathfrak{M}$ is perpendicular to the axis of the gyroscope.* Here, the component couple tending to produce rotation about the axis is nil, and hence the component angular velocity r about this axis† is constant throughout the motion; denote it by ν: $r = \nu$.

We are now prepared to state our conclusion in the restricted case.

THEOREM I. *The relation which holds between* κ, v, *and* $\mathfrak{F}$ *is the following*:

$$(A)\qquad \begin{cases} Av\dfrac{dv}{ds} = T, \\ A\kappa v^2 + C\nu v = Q, \end{cases}$$

where T *and* Q *denote respectively the components of* $\mathfrak{F}$ *along the positive tangent and the negative normal of* $\mathfrak{C}$ *at* ζ.

This theorem explains in part a vagueness of expression and ideas frequently met in discussions of the gyroscope, for questions are often asked, and answers given, both of which must, to have any meaning, involve all *three* quantities, κ, v, and $\mathfrak{F}$; but one is omitted. Thus, for example, the statement often made that "when a couple is applied to a rotating gyroscope, the forces of the couple intersecting the axis of the gyroscope at right angles, the axis will move in a plane perpendicular to the plane of the forces of the couple" is false. In fact, the axis will begin to move tangentially to this plane, if it starts from rest,‡ and all intermediate cases are possible, according to the initial motion of the axis.

The general case is that in which $\mathfrak{M}$ is arbitrary, O being fixed. Let the component of $\mathfrak{M}$ along $O\zeta$ be denoted by N. The other component of $\mathfrak{M}$, i. e., the component perpendicular to $O\zeta$, is dynamically equivalent to a force $\mathfrak{F}$ at ζ perpendicular to $O\zeta$ and equal numerically to this component. Let the components of $\mathfrak{F}$ along the positive tangent and negative normal of $\mathfrak{C}$ be denoted as before by T and Q. Then the complete statement of the dynamical conditions governing the motion is contained in the following theorem.

THEOREM II. *The three dynamical equations governing the motion of the gyroscope can be written as follows*:

* When $\mathfrak{M}$ is thought of as a couple, the *vector* $\mathfrak{M}$ is a vector at right angles to the plane of the couple, its length and sense being determined in the usual manner.

† By this is meant that the vector angular velocity about the instantaneous axis is resolved along three mutually perpendicular directions, one of which is the axis of the gyroscope; more precisely, the ray $O\zeta$. Moreover, the forces acting on the gyroscope may be replaced by a single force $\mathfrak{F}$ at ζ perpendicular to the axis, and the reaction at O. The plane of the couple $\mathfrak{M}$ is determined by $\mathfrak{F}$ and the point O, and the magnitude of $\mathfrak{M}$ is equal to that of $\mathfrak{F}$.

‡ The ordinary top with fixed peg is a case in point, if the top is released with the axis at rest.

$$(B) \qquad \begin{cases} Av\dfrac{dv}{ds} = T, \\ A\kappa v^2 + Crv = Q, \\ C\dfrac{dr}{dt} = N. \end{cases}$$

Equations (A) and (B) are of the nature of an intrinsic form of statement of the dynamical conditions. If T is a known function of v and s, the first equation gives v as a function of s. If, moreover, N is known in terms of v and s, the third equation, written in the form

$$Cv\frac{dr}{ds} = N,$$

yields r as a function of s. And now the second equation determines Q when κ is a known function of s. If, on the other hand, Q is a known function of v and s, it determines κ as a function of s. Here we have $\mathfrak{C}$ given in intrinsic form,

$$\kappa = \kappa(s).$$

Below is given the formula which expresses κ in terms of the latitude and longitude of ζ, and their derivatives. Hence the equation of $\mathfrak{C}$ in terms of these coördinates can be derived.

An advantage of the foregoing form of the equations of motion, invariant as it is of a particular choice of the usual coördinates, lies in the directness with which qualitative questions relating to the motion can be handled. The applications to the top which are considered below illustrate this feature; cf. in particular §6, in which the fallacious explanation which is commonly given for the rising of the top with blunt peg on a rough table is exposed.

Furthermore, in the cases of greatest interest, the above dynamical equations suffice to determine the motion of the axis of the gyroscope without bringing in Euler's geometrical equations and, in particular, the third Eulerian angle, φ; with which we are not at all concerned.

The proof of the general theorem (Theorem II) is simple. It consists in resolving the angular momentum of the system,—namely, the vector (σ),—along three mutually perpendicular moving axes, viz. (i) the axis of the gyroscope; (ii) the tangent to $\mathfrak{C}$, the path of ζ; (iii) the normal to $\mathfrak{C}$ in the tangent plane of the sphere. The vector form of the dynamical conditions governing the rotation is

$$\frac{d(\sigma)}{dt} = \mathfrak{M}.$$

The bending, κ, presents itself on differentiating the unit vector which lies along the above normal.

2. Proof of the Theorem

We recall the definitions and the fundamental theorem of rigid dynamics with respect to rotation.

Let a particle of mass m be moving with velocity represented by the vector $\mathfrak{v}$. Let O be a point fixed in space, and let $\mathfrak{r}$ be the vector drawn from O to the particle. The *angular momentum* of the particle with respect to O is defined as the vector product:

$$V\mathfrak{r}(m\mathfrak{v}).$$

For a system of discrete particles the angular momentum is defined as the sum of the angular momenta of the individual particles, and for a rigid body it is the corresponding integral. Let the latter be denoted by (σ). Then

$$(1)\qquad (\sigma) = \lim_{n\to\infty} \sum_{i=1}^{n} V\,\mathfrak{r}_i(\mathfrak{v}_i\,\Delta m_i).$$

Let a force represented by the vector $\mathfrak{F}$ act on m. The *moment* of $\mathfrak{F}$ with respect to O is defined as the vector product of $\mathfrak{r}$ and $\mathfrak{F}$:

$$V\mathfrak{r}\,\mathfrak{F}.$$

If a system of forces, $\mathfrak{F}_i$, act at points, P_i, of the body, the *moment* of the system with respect to O is defined as

$$\mathfrak{M} = \sum_{i=1}^{n} \mathfrak{r}_i\,\mathfrak{F}_i,$$

where $\mathfrak{r}_i$ is the vector drawn from O to P_i. In the case of a continuous distribution of force, the sum is to be replaced by the corresponding integral.

The fundamental theorem governing the rotation of a rigid body acted on by external, or applied, forces $\mathfrak{F}_i$ is

$$(2)\qquad \frac{d(\sigma)}{dt} = \mathfrak{M},$$

where $\mathfrak{M}$ relates only to these forces, and not to any internal forces.

Case I. We shall be concerned with this theorem in the case that a point of the axis of the gyroscope is fixed at O.

Case II. Let a rigid body be acted on by any applied forces, $\mathfrak{F}_i$, whatever. Let (σ) be computed, not with reference to a point O fixed in space, but with reference to the center of gravity, G, of the body; and let $\mathfrak{M}$ also be computed with

respect to G instead of O. Then equation (2) still holds, and governs the rotation of the body.

Case II is related to Case I as follows. Let a second body identical with the first have its center of gravity, G', fixed in space, and let it be acted on by the same *relative* forces; i. e., by forces of like intensities which are applied at corresponding points and act in corresponding directions, i. e., in the same directions relative to the second body. If, now, the two bodies were initially oriented alike and started alike, they will remain always oriented alike.

Since this latter body comes under Case I, it is sufficient for us to confine our attention to that case.

The vector (σ) can be expressed in terms of the moments of inertia of the body about three principal axes of inertia through O and the components of the vector angular velocity along these axes. Let three mutually perpendicular unit vectors be chosen as follows: $\mathfrak{a}$ shall be the vector drawn from O to ζ; $\mathfrak{t}$ shall be a

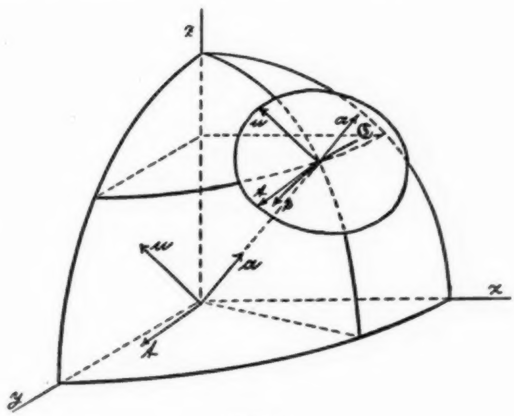

Fig. 1.

unit vector along the positive tangent to $\mathfrak{C}$; and $\mathfrak{n}$ shall be a unit vector along the positive normal to $\mathfrak{C}$. Let (ω) be the vector angular velocity of the body about the instantaneous axis, and let $\omega_{\mathfrak{t}}$, $\omega_{\mathfrak{n}}$, $\omega_{\mathfrak{a}}$ be the components of (ω) along axes through O parallel to or coincident with the above directions. Then, clearly, (for the definition of r cf. foot note in §1)

$$\omega_{\mathfrak{t}} = 0, \qquad \omega_{\mathfrak{n}} = v, \qquad \omega_{\mathfrak{a}} = r.$$

Moreover, the moments of inertia about these axes are:

$$I_{\mathfrak{t}} = A, \qquad I_{\mathfrak{n}} = A, \qquad I_{\mathfrak{a}} = C.$$

Now (σ) has, by a well known theorem, the value

$$(\sigma) = I_{\mathfrak{n}}\,\omega_{\mathfrak{n}}\,\mathfrak{n} + I_{\mathfrak{t}}\,\omega_{\mathfrak{t}}\,\mathfrak{t} + I_{\mathfrak{a}}\,\omega_{\mathfrak{a}}\,\mathfrak{a}.$$

Hence

(3) $$(\sigma) = Av\,\mathfrak{n} + Cr\,\mathfrak{a}.$$

From this equation it follows that

$$\frac{d(\sigma)}{dt} = Av\,\frac{d\mathfrak{n}}{dt} + Cr\,\frac{d\mathfrak{a}}{dt} + A\,\frac{dv}{dt}\,\mathfrak{n} + C\,\frac{dr}{dt}\,\mathfrak{a}.$$

It is clear that

$$\frac{d\mathfrak{a}}{dt} = v\,\mathfrak{t}.$$

Since $\mathfrak{n}$ is a unit vector normal to the cone $\mathfrak{K}$ at ζ, we see that $d\mathfrak{n}/ds$ must be a vector parallel to $\mathfrak{t}$ and having its length numerically equal to the bending, κ. Scrutiny of the sign shows that

$$\frac{d\mathfrak{n}}{dt} = \kappa v\,\mathfrak{t}.$$

Hence, finally,

(4) $$\frac{d(\sigma)}{dt} = Av\,\frac{dv}{ds}\,\mathfrak{n} + (A\kappa v^2 + Crv)\mathfrak{t} + C\,\frac{dr}{dt}\,\mathfrak{a}.$$

Let the vector $\mathfrak{M}$ be resolved along the same three directions:

(5) $$\mathfrak{M} = \mathfrak{M}_{\mathfrak{n}}\mathfrak{n} + \mathfrak{M}_{\mathfrak{t}}\mathfrak{t} + \mathfrak{M}_{\mathfrak{a}}\mathfrak{a}.$$

On equating the vectors (4) and (5) we obtain the three dynamical equations governing the rotation, in the form:

(6) $$Av\,\frac{dv}{ds} = \mathfrak{M}_{\mathfrak{n}}, \qquad A\kappa v^2 + Crv = \mathfrak{M}_{\mathfrak{t}}, \qquad C\,\frac{dr}{dt} = \mathfrak{M}_{\mathfrak{a}}.$$

The case in which we are particularly interested is that in which the component of $\mathfrak{M}$ perpendicular to the axis of figure is due to a force $\mathfrak{F}$ at ζ tangent to the sphere. In terms of the above unit vectors, $\mathfrak{F}$ has the value:

$$\mathfrak{F} = -Q\,\mathfrak{n} + T\,\mathfrak{t}.$$

Furthermore, $\mathfrak{M}_{\mathfrak{a}} = N$. We have, then, the three dynamical equations in the form:

$$Av\frac{dv}{ds} = T,$$

$$(7) \qquad A\kappa v^2 + Crv = Q,$$

$$C\frac{dr}{dt} = N. \qquad \text{q. e. d.}$$

3. The Bending, κ

The bending of a cone has been defined in §1 as the rate at which the tangent plane is turning when a point ζ on the generator at unit distance from the vertex moves with unit velocity.

Let

$$\theta = f(\psi)$$

be the equation of $\mathfrak{C}$, §1, where θ and ψ denote respectively the colatitude and the longitude of ζ, and let $f(\psi)$, together with its first and second derivatives, be continuous. Let $P : (\theta, \psi)$ be an ordinary point of $\mathfrak{C}$. If we denote by V the angle from the parallel of latitude through P with the sense of the increasing ψ to the tangent to $\mathfrak{C}$ at P with the sense of the increasing s, it is readily shown by means of infinitesimals that

$$\kappa = \frac{dV}{ds} - \frac{d\psi}{ds}\cos\theta.$$

Since

$$\tan V = \frac{d\theta}{d\psi \sin\theta}, \qquad \text{or} \qquad V = \tan^{-1}\frac{\theta'}{\psi' \sin\theta},$$

where accents shall denote differentiation with respect to s, and since the relation

$$ds^2 = d\theta^2 + d\psi^2 \sin^2\theta$$

gives

$$\theta'^2 + \psi'^2 \sin^2\theta = 1, \qquad \text{or} \qquad \psi' = \pm \csc\theta\sqrt{1 - \theta'^2},$$

it is found that

$$(1) \qquad \kappa = (\psi'\theta'' - \psi''\theta')\sin\theta - (1 + \theta'^2)\psi'\cos\theta.$$

Other formulas for κ are:

$$(2) \qquad \pm\kappa = \frac{\theta''}{\sqrt{1 - \theta'^2}} - \sqrt{1 - \theta'^2}\cot\theta;$$

$$\kappa = \frac{\dfrac{d^2\theta}{d\psi^2}\sin\theta - 2\dfrac{d\theta^2}{d\psi^2}\cos\theta - \sin^2\theta\cos\theta}{\pm\left[\dfrac{d\theta^2}{d\psi^2} + \sin^2\theta\right]^{\frac{3}{2}}}. \tag{3}$$

In all these formulas, the upper sign is to be taken when ψ increases with s.

If κ is given as a function of s, continuous together with its first derivative, formula (2) yields the following differential equation for determining θ:

$$\frac{d^2\theta}{ds^2} = \left(1 - \frac{d\theta^2}{ds^2}\right)\cot\theta \pm \kappa\left(1 - \frac{d\theta^2}{ds^2}\right)^{\frac{1}{2}}. \tag{4}$$

It is then possible to determine ψ from the equation

$$\frac{d\psi}{ds} = \pm\csc\theta\left(1 - \frac{d\theta^2}{ds^2}\right)^{\frac{1}{2}}. \tag{5}$$

It is readily shown by infinitesimals that

$$\mathfrak{t}' = \kappa\mathfrak{n} - \mathfrak{a},$$

where $\mathfrak{t}' = d\mathfrak{t}/ds$. Now, the absolute value of $\mathfrak{t}'$ is equal numerically to the curvature, K, of $\mathfrak{C}$ regarded as a twisted curve. Hence*

$$K^2 = \kappa^2 + 1. \tag{6}$$

Furthermore, the osculating plane of $\mathfrak{C}$ at ζ is determined by $\mathfrak{t}$ and $\mathfrak{t}'$, both drawn from ζ; and a unit vector, $\mathfrak{N}$, perpendicular to the osculating plane is given by the formula:

$$\mathfrak{N} = \frac{1}{\sqrt{1+\kappa^2}}\mathfrak{n} + \frac{\kappa}{\sqrt{1+\kappa^2}}\mathfrak{a}.$$

The torsion, T, of $\mathfrak{C}$ at ζ is equal numerically to the absolute value of $\mathfrak{N}'$ On computing $|\mathfrak{N}'|$ it is seen that†

$$\frac{d}{ds}\tan^{-1}\kappa = \pm T, \tag{7}$$

the formula found by Professor Haskins.

Let $\mathfrak{C}$ be referred to a system of Cartesian axes through O, and let $\mathfrak{i}$, $\mathfrak{j}$, $\mathfrak{k}$ be unit vectors along these axes. Then

* Cf. Coolidge, *Non-Euclidean Geometry*, p. 189, Theorem 2.

† Ibid., p. 193.

$$\mathfrak{a} = x\,\mathfrak{i} + y\,\mathfrak{j} + z\,\mathfrak{k},$$
$$\mathfrak{a}' = x'\,\mathfrak{i} + y'\,\mathfrak{j} + z'\,\mathfrak{k} = \mathfrak{t},$$
$$V\mathfrak{a}\mathfrak{a}' = (yz' - y'z)\,\mathfrak{i} + (zx' - z'x)\,\mathfrak{j} + (xy' - x'y)\,\mathfrak{k} = \mathfrak{n}.$$

We have seen that

$$\mathfrak{n}' = \kappa\,\mathfrak{t}.$$

Hence*

$$(8)\qquad \begin{aligned} yz'' - y''z &= \kappa x', \\ zx'' - z''x &= \kappa y', \\ xy'' - x''y &= \kappa z'. \end{aligned}$$

Since $|K| = |\mathfrak{t}'|$ and

$$\mathfrak{t}' = x''\,\mathfrak{i} + y''\,\mathfrak{j} + z''\,\mathfrak{k},$$

or

$$K^2 = x''^2 + y''^2 + z''^2,$$

we obtain, on squaring the equations (8) and adding, a new proof that

$$K^2 = \kappa^2 + 1.$$

If the cone $\mathfrak{K}$ is a cone of revolution, it follows at once from the definition of the bending that κ is constant. Let the axis of the cone pass through the north pole of the sphere, and let α be the semi-vertical angle of the cone. The equation of $\mathfrak{C}$ is $\theta = \alpha$, and it follows from (2) that

$$\pm\,\kappa = -\cot\alpha.$$

Suppose, conversely, that κ is constant. Then $\mathfrak{K}$ must be a cone of revolution.

A first proof is given by (7), for $\mathfrak{C}$ is now a curve whose torsion vanishes identically. It is an elementary fact of differential geometry that such a curve is a plane curve. And since $\mathfrak{C}$ lies on a sphere, it must be a circle.†

Another proof follows at once from (4). Let α be determined by the conditions:

$$\pm\,\kappa = -\cot\alpha, \qquad 0 < \alpha < \pi.$$

Let the north pole be so chosen that (i) a point, A, of $\mathfrak{C}$ will have the colatitude α; and (ii) $\mathfrak{C}$ will be tangent to the parallel of latitude at A. θ is determined

* Coolidge, loc. cit., p. 190.

† Professor Haskins applied his relation to this proof of the theorem.

uniquely as a function of s by (4) and the initial conditions:

$$\theta_0 = \alpha, \qquad \theta_0' = 0.$$

One solution satisfying these conditions is

$$\theta = \alpha;$$

and this is the only one. Hence the theorem is proved.

4. Applications

We will consider first some of the classical theorems about the gyroscope and show how their proofs follow immediately from the general or the restricted theorem of §1.

(*A*) *Steady Precession.* Let the axis of the gyroscope describe a cone of revolution at a constant rate, $\dot{\psi} = \dot{\psi}_0 > 0$, and let $N = 0$. Then $r = \nu$ (const.) > 0; T and Q are given by Theorem I and are:

$$T = 0, \qquad Q = C\nu v - Av^2 \cot \alpha, \tag{1}$$

where α is the semi-vertical angle of the cone; for

$$\kappa = -\cot \alpha.$$

The second equation (1) is the only relation which must hold between v, ν, α, and Q. In particular, $Q = 0$ if

$$C\nu = Av \cot \alpha. \tag{2}$$

If (i) $\quad \alpha > \cot^{-1} \dfrac{C\nu}{Av}, \quad$ then $\quad Q > 0;$

(ii) $\quad \alpha < \cot^{-1} \dfrac{C\nu}{Av}, \quad Q < 0.$

In Case (i) Q is directed away from the axis, i. e., along the outer normal of the cone; in Case (ii), the reverse.

Steady precession is also possible with $\dot{\psi} = \dot{\psi}_0 < 0$, $\nu > 0$. Here, $\kappa = \cot \alpha$, and

$$Q = C\nu\, v + Av^2 \cot \alpha. \tag{1'}$$

Thus Q is always directed *toward* the axis of the cone.

(*B*) *No Forces.* This case is completely covered by equation (2) above, provided $v \neq 0$:

$$C\nu = Av \cot \alpha. \tag{3}$$

Any two of the three quantities ν, v, α can be chosen at pleasure, subject merely to the restrictions:

$$\nu > 0, \quad v > 0, \quad 0 < \alpha < \frac{\pi}{2};$$

and the third is then uniquely determined by (3). If $\alpha = \pi/2$, then $\nu = 0$ and v is arbitrary.

Consider, for example, the earth, and suppose the sun, moon, and all other outside attracting matter were suddenly annihilated. What would be the effect on the motion of the axis, if the earth were rigid? A parallel to the axis through a fixed point describes a cone of revolution, of semi-vertical angle 23° 27′, once in 25,800 years, and the angular velocity, ν, corresponds to one revolution a day. The ratio C/A is about 305/306, and thus for present purposes is unity. Hence α can be computed from (3). It is found that the north pole would describe a circle of 21 inches diameter once a day (nearly).

Again, for the actual motion of the earth, the equation (1′) allows us to read off at once the value of the resultant couple exerted by the sun, the moon, and the other masses of the solar system.

(*C*) *The General Case*, $\kappa = 0$. Here ζ describes an arc of a great circle, with arbitrary velocity. Its motion along that circle is governed by the dynamical equations:

$$(4) \qquad A\frac{dv}{dt} = T, \qquad Crv = Q, \qquad C\frac{dr}{dt} = N.$$

The second of equations (4),

$$(5) \qquad Q = Crv,$$

is the one most strongly emphasized in accounts of the gyroscope, when v is constant and $N = 0$. But (5) is seen to hold regardless of the values of T and N. Thus it appears, for example, that sudden changes in T and N have no immediate effect on Q.

Again, if the motion of ζ in its path is prescribed; i. e., if v = a given function of the time or the space; then T will depend only on the time or the space, and not at all on r, which through proper choice of N may be made any function of the time whatever, whose derivative is continuous.

(*D*) *The Top*. A top with fixed peg, O, which is executing a steady precession, is a particular case of the motion studied under (*A*). Let α be the angle from the upward vertical through O to the positive axis, $O\zeta$, of the top, and let ψ be the longitude of the axis with reference to a vertical plane through O. Let the center of gravity lie on the positive axis, distant h from O. Then $\mathfrak{F}$ lies in the meridian plane and is tilted downward.

$$|\mathfrak{F}| = Mgh \sin \alpha = \pm Q, \qquad v = \pm \dot{\psi}_0 \sin \alpha, \qquad -\kappa = \pm \cot \alpha,$$

where the upper sign is to be chosen when $\dot{\psi}_0 > 0$; the lower, when $\dot{\psi}_0 < 0$. Thus we have from (1) the usual equation:

$$A\dot{\psi}_0^2 \cos \alpha - C\nu\dot{\psi}_0 + Mgh = 0. \tag{6}$$

When $\alpha = \pi/2$, (6) gives:

$$\dot{\psi}_0 = \frac{Mgh}{C\nu}. \tag{7}$$

For other values of α, (6) has a small positive root given approximately by (7), and a numerically large root algebraically slightly less than $C\nu/(A \cos \alpha)$.

(E) *The Grape-Vine.* Consider a top that is doing the grape vine.* Let it be mounted at its center of gravity, and let the only applied forces, aside from

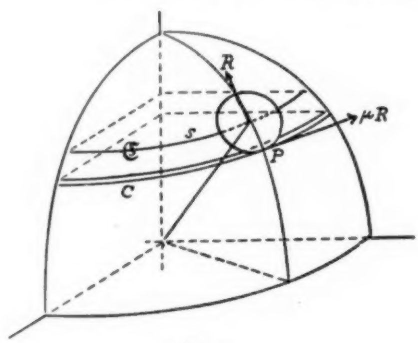

Fig. 2.

the reaction at the peg, be the reaction of the constraint, the curve C. This latter force may be taken as consisting of a component R normal to $\mathfrak{C}$ and the axis of the top and passing through ζ, and a component μR tangential to C at P and hence parallel to the tangent to $\mathfrak{C}$ at ζ, and lying in the tangent plane of the sphere at ζ. If we think of two forces equal and opposite to this latter component as applied at ζ, the whole system of applied forces yields:

$$T = \mu R, \qquad Q = R, \qquad N = -h\mu R,$$

where h is the radius of the axle of the top, or $P\zeta$.

We have assumed that initially r is large and v is positive and small, so that the axle is slipping on the constraint in the sense opposite to that of the motion of

* By this is meant the top of G. Sire, 1861. For the literature cf. Enzyklopädie der mathematischen Wissenschaften, vol. IV, p. 639.

$\mathfrak{z}$; moreover, that $\mathfrak{C}$ and C are related to each other in a one-to-one manner and continuously, so that, as $\mathfrak{z}$ describes $\mathfrak{C}$, P describes C in like sense.

Under these initial conditions the axle will press against the constraint, as we have assumed it to do. For, if the constraint were not there, i. e., if the gyroscope were given the above initial motion with ν large and v small, and left to itself, its axis would describe a very slender cone, since by (3) α would be small. Now, the constraint C is in the way of such a motion of the axis, and the best the axis can do is to follow a curve $\mathfrak{C}$ which is parallel to the constraint C.

So long as these conditions hold, the motion is governed by the equations of Theorem II, which here become:

$$(8) \qquad A\frac{dv}{dt} = \mu R, \qquad A\kappa v^2 + Crv = R, \qquad C\frac{dr}{dt} = -h\mu R.$$

From the first and third equations it appears that

$$Ah\frac{dv}{dt} + C\frac{dr}{dt} = 0,$$

$$(9) \qquad Ahv + Cr = Ahv_0 + Cr_0.$$

Since v is increasing and r decreasing, there will come an instant when either (a) slipping ceases or (b) the axle leaves the constraint.

In Case (a), if v_1, r_1 are the values v, r have when slipping ceases, then

$$v_1 = hr_1,$$

and hence, from (9),

$$(10) \qquad v_1 = \frac{Ah^2v_0 + Cr_0h}{Ah^2 + C}.$$

On eliminating R and r between the first two of equations (8), and (9), and setting $Ahv_0 + Cr_0 = AH$, we find:

$$(11) \qquad \frac{dv}{ds} = \mu(\kappa - h)v + \mu H.$$

Since κ is a function of s determined by the constraint, (11) is a linear differential equation of the first order for determining v as a function of s.

In Case (b), a first condition which must be realized, if the axle is to leave the constraint at an ordinary point P' of C, is that R shall vanish. Hence, from the second of equations (8),

$$-\kappa' = \frac{Cr'}{Av'} = \frac{H}{v'} - h.$$

At this instant there are no forces acting, and the axis of the gyroscope would like to describe a cone of revolution, for which $\kappa = \kappa'$. It will do so unless the constraint C is in the way. Construct the cone of revolution and denote its intersection with the sphere by $\mathfrak{C}_1$. Also continue C beyond P' and denote the path ζ would describe if the axle did not leave the constraint by $\mathfrak{C}_2$. If $\mathfrak{C}_2$, near P', lies inside $\mathfrak{C}_1$, the axle will leave the constraint; if $\mathfrak{C}_2$ lies outside of $\mathfrak{C}_1$ or coincides with it, the axle will not leave the constraint.

Suppose the curve C comes to an end at a point $\bar{P}$, but that up to and inclusive of $\bar{P}$ is related in a one-to-one manner and continuously to its parallel curve $\mathfrak{C}$, and that the axle has not left the constraint, but reaches $\bar{P}$. Will the axle leave C at $\bar{P}$? The answer can be given as in the previous case. Let $\bar{v}$ be the value of v at $\bar{P}$, and let $\bar{\kappa}$ be the bending of the cone the axis would now describe if the restraint were removed. By (3)

$$-\bar{\kappa} = \cot \bar{\alpha} = \frac{C\bar{r}}{A\bar{v}}.$$

On the other hand, if the axle clings to $\bar{P}$, the axis will describe a cone whose semi-vertical angle is

$$\beta = \tan^{-1} h.$$

The condition that the axle should cling to $\bar{P}$ is obviously that $\bar{\alpha} \leqq \beta$, or

$$A\bar{v} \leqq Ch\bar{r}. \tag{12}$$

I have discussed this problem at some length because it illustrates the ease with which quantitative results can be obtained by the theorem of §1.

(*F*) *The Centrifugal Mill.* The Griffin Mill described by Webster* is a particular case of the problem just discussed if gravity is neglected and the mill set in motion under the action of no other forces than the natural constraints. Since κ is here constant, equation (11) can be integrated in closed form and the motion discussed in all detail.

Doubtless, however, to make the mill meet the needs of practice, the angular velocity r would be controlled by a motor which would exert a couple just sufficient to offset the above couple μhR and thus render r constant: $r = \nu$. There would then be a further tangential force applied at ζ, which would control the velocity v and make it constant: $v = v_0$. The second equation of Theorem I would then determine $Q = R$:†

$$R = A\kappa v_0^2 + C\nu v_0, \tag{13}$$

* *Dynamics*, 2d edition, Teubner, 1912, p. 273.

† The result agrees with Webster's formula (63), p. 274. Webster resolves the angular velocity about the instantaneous axis along two *oblique* directions.

where $\kappa = \cot\alpha$ and α is the angle the axis of the shaft makes with the downward vertical,—in the figure, the angle that $O\mathfrak{z}$ makes with the upward vertical. κ is here positive because $\mathfrak{C}$ lies to the left of the positive tangent.

(*G*) *Friction at* $\mathfrak{z}$. It is interesting to discuss a case in which the force $\mathfrak{F}$ at $\mathfrak{z}$ is always tangent to the path, $\mathfrak{C}$. Let the gyroscope be encased in a hollow sphere with centre at O and radius slightly greater than 1; and let it carry a brush at $\mathfrak{z}$ which rubs against the shell, thus exerting a constant force f along the negative tangent at $\mathfrak{z}$. Also, let $N = 0$. The dynamical equations thus become:

$$(14)\qquad Av\frac{dv}{ds} = -f, \qquad A\kappa v^2 + C\nu v = 0, \qquad N = 0.$$

From the first equation,

$$v^2 = v_0^2 - \frac{2f}{A}s,$$

$$v = \frac{ds}{dt} = \sqrt{v_0^2 - \frac{2f}{A}s},$$

$$t = \frac{A}{f}\left\{v_0 - \sqrt{v_0^2 - \frac{2f}{A}s}\right\}.$$

Hence $\mathfrak{z}$ comes to rest after travelling a distance $s = Av_0^2/2f$ in time $t = Av_0/f$.

The curvature, κ, is given by the equation:

$$\kappa = -\frac{C\nu}{Av}.$$

As v steadily decreases, κ increases numerically without limit, and the curve $\mathfrak{C}$ bends ever more sharply to the right. θ is given as a function of s by (4), §3.

It is to be observed that in this example the motion violates throughout the rule often given as the alpha and omega of the gyroscope,—to wit, that the axis moves at right angles to the forces of the couple, or the force $\mathfrak{F}$. Here, the point $\mathfrak{z}$ always moves precisely in the line of the applied force, $\mathfrak{F}$.

One can imagine a case in which $\mathfrak{F}$ is just the reverse of that in the foregoing example. A centipede is forced to crawl on the inner surface of the shell and always draw the point $\mathfrak{z}$ after him. The Devil sits at $\mathfrak{z}$ and drives the centipede in the direction of the positive tangent. Here, $T = f > 0$, $Q = 0$, $N = 0$, and a discussion similar to the above is in place. κ now approaches 0 as v increases without limit, and it is a matter of conjecture as to whether $\mathfrak{C}$ has an asymptotic great circle.

5. The Top with Fixed Peg

The problem of the top with fixed peg, spinning with high angular velocity about its axis and acted on by gravity, but not by friction, is treated quantitatively by elementary methods, the latitude and longitude of $\mathfrak{z}$ being obtained as functions of the time by means of quadratures.* What these formulas fail to show is that the cone $\mathfrak{K}$ has the least number of inflectional tangent planes which is possible for any particular case of motion. This gap is easily filled by the foregoing methods, which furthermore yield an essential simplification in setting up the differential equations that govern the motion. It is found that the dynamical equations alone are sufficient to determine the motion of the axis, and thus all use of Euler's geometrical equations is avoided. In point of simplicity and directness, therefore, the following treatment would seem to yield a real gain.

Let θ and ψ be respectively the colatitude and the longitude of ζ, the ray $\theta = 0$ being the upward vertical through the peg, O.

It is clear that the force $\mathfrak{F}$ is directed down the meridian through ζ, and that its magnitude is $Mgh \sin \theta$, where h denotes the distance from O to the center of gravity, assumed to lie on the same side of O as ζ. Hence the components T and Q have the values:

$$(1) \qquad T = Mgh \sin \theta \frac{d\theta}{ds}, \qquad Q = Mgh \sin^2 \theta \frac{d\psi}{ds}.$$

The value of N is 0, and so $r = \nu$.

From Theorem I we have:

$$(2) \qquad Av \frac{dv}{ds} = Mgh \sin \theta \frac{d\theta}{ds}.$$

Integrating and setting

$$u = \cos \theta, \qquad a = \frac{2Mgh}{A},$$

we obtain, on letting the subscript refer to initial values:

$$(\text{i}) \qquad v^2 = v_0^2 + a(u_0 - u).$$

It is obvious that this is in substance the equation of energy.

A second integral of the dynamical equations is obtained by observing that, since $\mathfrak{M}$ is always horizontal, and since

$$\frac{d(\sigma)}{dt} = \mathfrak{M},$$

* For the classical treatment cf. Appell, *Mécanique rationnelle*, vol. 2, 2d edition, p. 195.

the vertical component of (σ) is constant.* Now, by §2, (3),

$$(\sigma) = Av\,\mathfrak{n} + Cr\,\mathfrak{a}.$$

The component in question has, therefore, the value:

$$Av\left(\frac{d\psi}{ds}\sin\theta\right)\sin\theta + C\nu\cos\theta.$$

Hence we obtain as the integral in question:

$$Av\frac{d\psi}{ds}\sin^2\theta + C\nu\cos\theta = \text{const.} \tag{3}$$

On setting

$$\epsilon = \dot{\psi}_0(1 - u_0^2), \qquad \gamma = \frac{C\nu}{A}, \tag{4}$$

equation (3) becomes:

$$v(1 - u^2)\frac{d\psi}{ds} = \epsilon + \gamma(u_0 - u). \tag{ii}$$

Finally, there is the geometric relation arising from the equation

$$ds^2 = d\theta^2 + d\psi^2\sin^2\theta,$$

namely:

$$\left(\frac{d\theta}{ds}\right)^2 + \left(\frac{d\psi}{ds}\right)^2(1 - u^2) = 1. \tag{iii}$$

From equations (i), (ii), and (iii) the two differential equations for θ and ψ in the usual form follow at once:†

$$\begin{aligned}\left(\frac{du}{dt}\right)^2 &= (u_0 - u)(u - u_1)(\lambda u + \mu),\\ \frac{d\psi}{dt} &= \frac{\epsilon + \gamma(u_0 - u)}{1 - u^2},\end{aligned} \tag{5}$$

* This remark is standard for obtaining one of the differential equations of motion of the top. The form of (σ) given just below leads immediately to this equation (3).

† The explicit formula obtained by elimination for $(du/dt)^2$ involves a cubic polynomial:

$$\frac{du^2}{dt^2} = f(u) = (1 - u^2)\;[v_0^2 + a(u_0 - u)] - [\epsilon + \gamma(u_0 - u)]^2.$$

Since $f(1) \leqq 0$ and $f(+\infty) = +\infty$, $f(u)$ has one root $\geqq +1$. Furthermore, $f(u)$ has a second root, u_0, between -1 and $+1$. Finally, $f(-1) \leqq 0$. Hence, considering the character of the graph of such a cubic, we see that $f(u)$ must have its third root, u_1, in the interval $-1 \leqq u \leqq 1$.

where u_1, λ, μ are constants such that $-1 \leqq u_1 \leqq -1$, and $\lambda u + \mu \neq 0$ in the interval $-1 < u < 1$; moreover, $\dot{\theta}_0 = 0$. We prefer, however, to write these equations in the form:

$$(6) \qquad \begin{aligned} \left(\frac{d\theta}{ds}\right)^2 &= \frac{(u_0 - u)(u - u_1)(\lambda u + \mu)}{v^2(1 - u^2)}, \\ \frac{d\psi}{ds} &= \frac{\epsilon + \gamma(u_0 - u)}{v(1 - u^2)}, \end{aligned}$$

where v is given by (i), and $u = \cos\theta$. Equations (i) and (5) show that s can be expressed as a function of u by a quadrature. On substituting the inverse function for u in (6), θ and ψ can be expressed as functions of s by means of quadratures.

We next compute κ. From (1) and (6),

$$Q = \frac{Mgh}{v}\left\{\epsilon + \gamma(u_0 - u)\right\}.$$

On substituting this value in the second equation of Theorem I and solving for κ, we have:

$$(\text{iv}) \qquad -\kappa = \frac{C\nu}{Av^3}\left\{ v_0^2 - \frac{Mgh\epsilon}{C\nu} + \frac{a}{2}(u_0 - u)\right\}.$$

We now have all the material in hand for a detailed discussion of the character of $\mathfrak{C}$ in each of the three cases indicated by Fig. 3. In each case, $\dot{\theta}_0 = 0$. Moreover, in

Case I, $0 < \dot{\psi}_0 < c$; Case II, $\dot{\psi}_0 = 0$; Case III, $-\dfrac{\gamma}{1 - u_0} < \dot{\psi}_0 < 0$.

I II III

Fig. 3.

Here, c is the value of $\dot{\psi}_0$ corresponding to a steady precession and is the numerically smaller of the two roots of (6), §4.

Case I. The initial value of κ is given by the equation:

$$(7) \qquad -\kappa_0 = \frac{C\nu}{Av_0^3}\left(v_0^2 - \frac{Mgh\epsilon}{C\nu}\right) = \frac{\gamma}{\sqrt{1 - u_0^2}}\left\{\frac{1}{\dot{\psi}_0} - \frac{a}{2\gamma}\cdot\frac{1}{\dot{\psi}_0^2}\right\}.$$

For values of $\dot{\psi}_0$ in the interval $0 \leqq \dot{\psi}_0 \leqq c$, the derivative

$$\frac{d(-\kappa_0)}{d\dot{\psi}_0} = \frac{\gamma}{\dot{\psi}_0^3\sqrt{1 - u_0^2}}\left\{\frac{a}{\gamma} - \dot{\psi}_0\right\}$$

is positive; for, the value of c is approximately, by §4, (7), $a/(2\gamma)$. Hence the derivative is positive for $\dot{\psi}_0 = c$; and for smaller values of $\dot{\psi}_0$, the brace is still larger.

Thus a positive value of $\dot{\psi}_0$ less than c gives rise to an algebraically larger value of κ_0 than the one which corresponds to c, and $\mathfrak{C}$ begins to sink below the initial parallel of latitude, $\theta = \theta_0$. It follows, then, that θ increases, and hence $u_1 < u_0$. Moreover, $u_1 > -1$; for $f(-1) < 0$.

From equations (6) it is now evident that $\mathfrak{C}$ passes from the parallel of latitude $\theta = \theta_0$ to $\theta = \theta_1$, both θ and ψ steadily increasing with s, and that this arc is tangent to each parallel at an extremity. Moreover, it is clear that the cone $\mathfrak{K}$ must have an inflectional tangent plane at some point of this arc. What is not clear, however, from equations (6) is that there is *only one* such plane. This fact appears at once from (iv). The condition for an inflectional tangent plane is $\kappa = 0$, or

$$(8) \qquad v_0^2 - \frac{Mgh\epsilon}{C\nu} + \frac{a}{2}(u_0 - u) = 0.$$

This equation must have at least one root, and it is seen at once that it has only one. Moreover, (8) gives the explicit determination of u for this plane:

$$(9) \qquad u = u_0 + \frac{2}{a}\left(v_0^2 - \frac{Mgh\epsilon}{C\nu}\right).$$

The remainder of $\mathfrak{C}$ is found by reflection of the arc just obtained in the meridian plane through its lowest point, and periodic repetitions of the complete arc thus constructed.

Case II. Here, $\dot{\psi}_0 = 0$ and so $\epsilon = 0$. The point ζ starts from rest, and since $v^2 = a(u_0 - u)$ from (i), it follows from (6) that $u_1 < u_0$. Hence θ increases. Moreover, $u_1 > -1$, as in Case I. The discussion is similar to that of Case I. It is clear from (iv) that κ is never 0 or positive, and hence the cone $\mathfrak{K}$ has no inflectional tangent planes. It has, of course, sharp edges at the cusps of $\mathfrak{C}$.

Case III. Here, $\dot{\psi}_0 < 0$, $\epsilon < 0$. From (iv) it appears that κ is initially negative,* and hence θ begins by increasing. Thus $u_1 < u_0$ and θ steadily increases till $\theta = \theta_1$. Consequently each term in the brace of (iv) is positive, and hence κ is negative along the whole arc. Thus the cone $\mathfrak{K}$ has no inflectional tangent plane. From (6) it follows that, at the point C where $\mathfrak{C}$ meets the parallel of latitude $\theta = \theta_1$, $\mathfrak{C}$ is tangent to this circle.

For values of $\dot{\psi}_0$ which are numerically small, ϵ will also be numerically small,

* We point out the inference to be drawn from this fact, namely, that near the initial point $\mathfrak{C}$ lies below the great circle arc which is tangent to $\mathfrak{C}$ at that point.

and the second bracket in $f(u)$ will vanish for a value of u lying in the interval $u_1 < u < u_0$:

$$u' = u_0 + \frac{\epsilon}{\gamma}.$$

For this value of u, $d\psi/ds$ changes sign, and ψ increases with s in the interval $u_1 \leqq u < u'$.

On the other hand, if $\dot\psi_0$, and hence ϵ, is numerically large, that bracket will never vanish, and ψ will decrease forever. Obviously, in this case, too, κ is always negative. The value of ϵ which separates these two cases is given by the condition that that bracket shall vanish for $u = u_1$. Consequently u_1 here $= -1$, and we have:

$$\epsilon + \gamma(u_0 + 1) = 0, \qquad \dot\psi_0(1 - u_0^2) + \gamma(u_0 + 1) = 0,$$

$$\dot\psi_0 = -\frac{\gamma}{1 - u_0}.$$

Hence the second inequality for Case III.

If $\dot\psi_0 = -\gamma/(1 - u_0)$, ζ passes through the south pole of the sphere.

Under Case I we demanded that $\dot\psi_0 < c$. If $\dot\psi_0 > c$, the curve $\mathfrak{C}$ rises above the parallel of latitude $\theta = \theta_0$. For values of $\dot\psi_0$ only slightly greater than c, we get curves $\mathfrak{C}$ like those in Case I, the parallel $\theta = \theta_0$ playing the rôle of $\theta = \theta_1$ in Case I. As $\dot\psi_0$ continues to increase, we get curves $\mathfrak{C}$ like those of Case III, the intermediate case being characterized by the fact that the root u_1 of $f(u)$ (which now is $> u_0$) must be the root of (8). The cases which correspond to Case III are those for which the second bracket in $f(u)$ has a root in the interval $u_0 < u < u_1$. When $\dot\psi_0$ has increased sufficiently, $d\psi/ds$ is always > 0. The two cases are separated by the value of $\dot\psi_0$ for which that bracket and $f(u)$ vanish simultaneously. Here, $u_1 = 1$, ζ goes through the north pole, and

$$\epsilon + \gamma(u_0 - 1) = 0, \qquad \dot\psi_0(1 - u_0^2) + \gamma(u_0 - 1) = 0,$$

$$\dot\psi_0 = \frac{\gamma}{1 + u_0}.$$

When $\dot\psi_0 > \gamma/(1 + u_0)$, κ is always negative, and the path of ζ touches alternately the circles $\theta = \theta_0$ and $\theta = \theta_1$, which coalesce when $\dot\psi_0$ is equal to the larger root of (6), §4. A similar remark applies in Case III when $\dot\psi_0 < -\gamma/(1 - u_0)$.

Concerning Case III, we mention the one important fact which we have not been able to establish by the present methods,—namely, that $\psi_1 > \psi_0$. Hadamard has, however, given an altogether simple proof of this relation by means of Cauchy's Integral Theorem; cf. Bulletin des Sciences Mathématiques, vol. 30 (1895), p. 228.

6. The Top with Blunt Rough Peg

Consider a top with blunt peg spinning on a smooth horizontal plane. The peg is not approximately a conical point, but may be thought of as hemispherical, or as having the form of any surface of revolution which would correspond to a real top. The top is given a high angular velocity about its axis and released, or started with a moderate velocity of the axis, the peg touching a smooth horizontal table. We shall, furthermore, assume that the peg remains throughout the motion in contact with the table.

Since the forces acting, gravity and the reaction of the table, are both vertical, the horizontal motion of the center of gravity, G, will be uniform, and it is convenient to think of the initial velocity of G as having no horizontal component. G will then remain permanently in the vertical drawn through its initial position.

The cone $\mathfrak{K}$ is here the family of rays drawn from an arbitrary fixed point, O, in space parallel to the axis of the top.

It is easy to set up the equations of motion, for the problem is closely akin to that of the top with fixed peg. The motion of the axis is similar to that of the earlier case. The axis falls and rises periodically, and executes a precession.* In particular, let the top be started so that we have Case I,—cf. Fig. 3, §5.

A Rough Spot. Suppose the peg, in its course over the table, comes to a rough spot. Suppose, furthermore, that the point P of the peg in contact with the

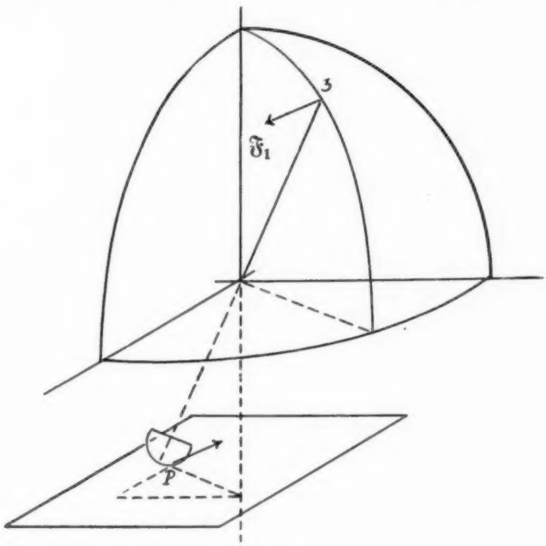

Fig. 4.

* Cf. Appell, loc. cit., p. 234.

table is moving in a direction nearly or quite at right angles with the axis and toward the left of an observer looking from G toward the peg. These conditions will be fulfilled since the motion of the axis is slow and the angular velocity of the top about its axis is high, if the peg be not too diminutive and the angle of the axis with the vertical reasonably large.

The horizontal force exerted by the table is here nearly or quite perpendicular to the axis of the top and toward the right of the observer. Let two forces, one equal and the other opposite to this force, be thought of as applied at the point R of the axis nearest P.

To study the rotation of the actual top we will introduce a second top equal to the first, with its center of gravity fixed at O. It shall be oriented initially like the first and have initially the same (vector) angular velocity. Finally, it shall be acted on by the same relative forces as those which act on the actual top. It will, then, permanently be oriented like the actual top. The motion of the second top is governed by the theorems of §1.

These new forces, the force of friction at P and the two equal and opposite forces at R, give rise, so far as the rotation of the second top is concerned, to two new couples, namely: (*a*) a couple about the axis of the top, which diminishes the component of the angular velocity along that axis; its numerical value is small, and it has relatively small effect till considerable time has elapsed; (*b*) a couple represented by a force $\mathfrak{F}_1$ at ζ perpendicular to the axis of the top and nearly or

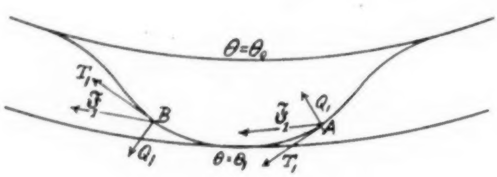

Fig. 5.

quite perpendicular to the meridian plane through ζ, its sense corresponding to that in which this plane is rotating. Let its components along the tangent and normal be T_1 and Q_1, and let T and Q be the components of the force $\mathfrak{F}$ due to gravity.

Will the effect of the friction be to make the axis of the top rise or fall? We are now in a position to answer this question.

Case A. Suppose that, when the peg came to the rough spot, the axis was descending, ζ being at the point A of $\mathfrak{C}$, Fig. 5. The value of κ just before A was reached was, by Theorem II, §1:

$$\kappa = \frac{Q}{Av^2} - \frac{Cr}{Av}. \tag{1}$$

Just after A is reached, κ has the value

$$\bar{\kappa} = \frac{Q + Q_1}{Av^2} - \frac{Cr}{Av} = \kappa + \frac{Q_1}{Av^2}. \tag{2}$$

There is no sudden change in the values of v and r near A; but the normal component of the applied force has jumped from Q to $Q + Q_1$, and the bending of the cone $\mathfrak{K}$ has thus experienced a discontinuity given by equation (2). Since Q_1 is negative, $\bar{\kappa}$ is less than κ, and so the new cone bears more sharply to the right (or less sharply to the left); thus the axis rises above the positions it would assume for like values of the longitude if the table were smooth.

Case B. Suppose, on the other hand, that, when the peg comes to the rough spot, the axis is ascending, ζ being at the point B of $\mathfrak{C}$. Equations (1) and (2) still hold just before and just after B is reached. Again there is a discontinuity in the bending. But here Q_1 is positive, $\bar{\kappa}$ is greater than κ, and the new cone bears more sharply to the left (or less sharply to the right). Hence the axis falls below the positions it would assume for like values of the longitude if the table were smooth.

The inference to be drawn from this example is that the rising of the axis of the top with rough peg is essentially a problem *im Grossen.* The explanations of the rising which depend on considerations *im Kleinen* are erroneous.* It can be shown, it is true, that both in Case A and in Case B the angular momentum, i. e., the vector (σ), rises. But the rising of (σ) is neither a necessary nor a sufficient condition *im Kleinen* for the rising of the axis of the top.

Top with Fixed Peg, under Damping. Consider a gyroscope with a point O of the axis fixed, which is acted on by no forces save the damping of the atmosphere and friction. Since the effect of the damping is here mainly to retard the angular velocity about the axis, we obtain a good approximation to the physical situation by assuming $\mathfrak{F} = 0$, and N negative and numerically small. Since $T = 0$, the velocity of the point ζ in its path, $\mathfrak{C}$, is constant:

$$\frac{ds}{dt} = v_0, \qquad s = v_0 t.$$

Furthermore, since $Q = 0$,

$$-\kappa = \frac{Cr}{Av_0}, \tag{3}$$

and the cone κ steadily becomes flatter.

Finally,

$$C\frac{dr}{dt} = N, \qquad r = r_0 - ct,$$

* Cf. Webster, loc. cit., p. 304.

where $c = -N/C$. Hence

$$r = r_0 - \frac{c}{v_0} s,$$

and κ is seen from (3) to be a linear function of s:

$$\kappa = -\frac{C}{Av_0}\left(r_0 - \frac{c}{v_0} s\right).$$

On substituting this value of κ in the differential equation for θ as a function of s, §3, (4), and solving, we have, with the further aid of equation (5), §3, the analytic representation of $\mathfrak{C}$ in this case.

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THE RELATIVE DISTRIBUTION OF THE REAL ROOTS OF A SYSTEM OF POLYNOMIALS*

BY

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1. Introduction

The problem about to be discussed may be looked upon as a generalization of the classical problem solved by Sturm in 1829 with regard to the real roots of a polynomial.† In Sturm's theorem it is shown how the number of distinct roots of a single polynomial which fall within a given real interval may be determined through a process rational in the coefficients. In the present paper we shall study a system of two or more polynomials in a single variable, and our aim will be to develop a rational process by which the order of succession of the roots of the several polynomials in a given real interval may be discovered.

Let us confine ourselves, at least for the present, to the case where the end-values of the interval are not roots and where the polynomials have only simple roots in the interval. It is clear that in such a case Sturm's theorem determines the only relations of the real roots of a single polynomial to the interval that remain invariant under continuous transformation of the real number system into itself. A similar remark applies to the theory about to be developed with respect to the real root system of several polynomials; so that we are undertaking the study of a problem which, from the point of view of a one-dimensional analysis situs, may be said to be the fundamental problem of the system under consideration.

The greater part of the paper will be devoted to the case of two polynomials. If the roots of the first are denoted generally by α and those of the second by β, and if we write down the roots within the interval considered in increasing numerical order (as $\alpha\alpha\alpha\beta\beta\alpha\beta\beta\beta\alpha\beta$), the β's effect a certain partition of the α's (in the present case into groups of 3, 0, 1, 0, 0, 1, 0,). The solution of the problem will consist in the determination of the numbers of α's in the successive groups.

* Presented to the Society, September 7, 1920.

† Bulletin de Férussac, vol. 11 (1829); Mémoires des Savants étrangers, vol. 6 (1835).

Previous writers have considered a less precise form of the problem, namely that in which α- and β-groups are counted only modulo 2. Cauchy* in 1837 or earlier examined the relation of the remainders in the highest common factor process for two polynomials to the changes in sign of these polynomials, thus obtaining essentially the solution of our problem modulo 2. Sylvester† in 1853 gave to these relations the interpretation in which we are here interested, in connection with the ordinal distribution of the roots. In Sylvester's paper the arrangement of α's and β's is called an "intercalation scale." The arrangement obtained by removing pairs of consecutive α's or β's as long as any such pairs are present is called an "effective intercalation scale." Thus, in the above example, the arrangement reduces, by cancellation of the α-pairs, to $\alpha\beta\beta\alpha\beta\beta\beta\alpha\beta$; then, by removal of β's, to $\alpha\alpha\beta\alpha\beta$; and finally the effective intercalation scale is $\beta\alpha\beta$. This reduced arrangement is connected by Sylvester with the signs that the two functions and their Sturmian remainders take at the ends of the interval.‡ A generalization of the theory to the case of n functions in $(n-1)$ variables, proposed by Sylvester,§ was carried out by Kronecker|| in 1869, in his theory of the "characteristic," which determined, also to modulus 2, the relative arrangement of the intersections of one closed curve in $(n-1)$-space with two hypersurfaces. Neither these nor other writers appear to have considered the problem of the complete root-distribution, whether in one or more variables.¶ It is the object of the present paper to obtain a solution for the one-dimensional case, to make some generalizations, and to discuss a number of special points in connection with the theory.

The solution of the problem for two polynomials will be found in § 3, an alternative method being given also, in § 13. In both methods use is made of the sequence of Sturmian functions derived from one of the polynomials. In the one case these serve as material for a number of additional Sturmian processes. In the alternative method they are used in building up a number of quadratic forms on whose signatures the solution depends.

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* *Calcul des indices des fonctions*, Journal de l'Ecole polytechnique, vol. 15 (1837), pp. 176–229. Part of the theory was developed in a memoir presented to the Turin Academy in 1831.

† *A theory of the syzygetic relations of two rational integral functions, comprising an application to the theory of Sturm's functions and that of the greatest algebraical common measure*, Philosophical Transactions of the Royal Society of London, vol. 143, part 3 (1853), pp. 407–548; *Collected Mathematical Papers*, vol. 1, pp. 429–586.

‡ Ibid., Art. 48–50; (*Collected Mathematical Papers*, vol. 1, pp. 517–521).

§ Ibid., Art. 55; (*Collected Mathematical Papers*, vol. 1, pp. 527–8).

|| *Ueber Systeme von Functionen mehrer Variabeln*, Berliner Monatsberichte, 1869, pp. 159–183, 688–697.

Ueber die Charakteristik von Functionen-Systemen, ibid., 1878, pp. 145–152.

¶ See also Cayley, *On the geometrical presentation of Cauchy's theorems of root-limitation*, Transactions of Cambridge Philosophical Society, vol. 12, part 2 (1877), pp. 395–413; *Collected Mathematical Papers*, vol. 9, pp. 21–39.

In § 7 will be found the theory for a system of more than two polynomials.

In § 8 the simple interval previously used will be replaced by the set of intervals defined by a system of algebraic inequalities; and in § 9 will be discussed a point arising out of this notion, whose chief interest consists in the connection that it makes with the restricted theory of Cauchy and Sylvester.

Since the reader may feel some disappointment at the laborious character of the computations demanded in numerical applications of the theory, it seems proper to include (in §§5, 10, and 12) a number of remarks tending to mitigate this condition. Even so, however, the main interest of the problem must remain theoretical; nor does it seem likely that any method could be devised capable of rapid application to numerical cases.

Among a variety of special problems to which § 11 will be devoted, we may notice in particular the determination of the number of separate intervals of the real number-system defined by a set of real algebraic inequalities. Two solutions will be given for this problem (in vii and viii).

2. The Cauchy-Sylvester theory

It will be useful to have before us the existing theory of the reduced root-distribution in a form that can be readily applied in obtaining a complete theory.

Let a and b be two real numbers, $a < b$. Let $f_0(x)$ be a polynomial with real coefficients, which does not vanish when $x = a$ or b, and which has in the interval (a, b) no multiple roots and no roots in common with another given real polynomial $\varphi(x)$. Let $f_1(x)$ be the first derivative of $f_0(x)$. Let those real roots of $f_0(x)$ which lie in (a, b) be $\alpha_1, \alpha_2, \ldots$

Let $S(f_0, \varphi)$ denote the Sturmian sequence $[f_0(x), \varphi(x), \varphi'(x), \varphi''(x), \ldots]$ obtained as follows: $-\varphi'(x)$ is the polynomial found as remainder when f_0 is divided by φ, $-\varphi''$ the remainder when φ is divided by φ', and so on, the last function in the sequence being the highest common factor of f_0 and φ. Let $V(x)$ be the number of variations of sign between consecutive elements of the sequence $S(f_0, \varphi)$ for any given x.

The Cauchy-Sylvester theorem may then be stated in the following form: *the number of α's making the quotient $\varphi(\alpha)/f_1(\alpha)$ positive exceeds the number of α's making it negative by $V(a) - V(b)$.*

This excess is called by Cauchy the "index" of f_0 and φ in (a, b).

It is easy to see that, with a suitable convention as to sign, the index enumerates the α's in the reduced or effective intercalation scale, and it is in such a form that Sylvester obtains the theorem.* It is, however, a simple matter to

* For a physical interpretation of the effective intercalations in terms of two interlacing strings, see Sylvester, *Sur l'entrelacement d'une fonction par rapport à une autre*, Journal für Mathematik, vol. 88 (1880), pp. 1–3; *Collected Mathematical Papers*, vol. 3, pp. 449–450.

prove the rule in its present shape by studying the continuous variation of x, and by observing that (i) the last function of the sequence vanishes for no x in (a, b), (ii) when any other function of the sequence, except $f_0(x)$, vanishes, the two adjacent functions have opposite signs, and (iii) when $x = \alpha + \delta$, $f_0(\alpha)$ being zero, $f_0(x)/f_1(x)$ has the same sign as δ for small enough values of $|\delta|$.

An immediate corollary of this theorem will be of use to us a little later. With the same assumptions, we may replace $\varphi(x)$ by the product of $\varphi(x)$ by $f_1(x)$. Hence *the number of α's making $\varphi(\alpha)$ positive exceeds the number making it negative by $U(a) - U(b)$, where $U(x)$ is the number of variations of sign in $S\{f_0(x), \varphi(x)\cdot f_1(x)\}$.*

3. Complete theory for two polynomials

Let a and b be real numbers of which $a < b$. Let $f_0(x)$ and $g_0(x)$ be given polynomials with real coefficients, and $f_1(x)$, $g_1(x)$ their first derivatives. Let us denote the sequence of Sturmian functions $S(g_0, g_1)$ by $g_0(x)$, $g_1(x)$, $g_2(x)$, $\ldots$, $g_k(x)$.* Let us assume that (i) neither a nor b is a root of $f_0(x)$, (ii) $f_0(x)$ has no multiple root in (a, b), (iii) $f_0(x)$ has no root in (a, b) in common with any one of $g_0(x)$, $g_1(x)$, $\ldots$, $g_k(x)$. Suppose the roots of $f_0(x)$ in (a, b) to be α_1, α_2, $\ldots$, and suppose the distinct roots of $g_0(x)$ in the interval $a < x \leqq b$ to be β_1, β_2, $\ldots$.† Our object is to show how the partition of the α's by the β's may be determined.

We naturally inquire how many of the α's give to the sequence $g_0(\alpha)$, $g_1(\alpha)$, $\ldots$, $g_k(\alpha)$ some particular number r of variations of sign. For, if we knew this variation number r for a certain α, we should be able to identify, by using Sturm's theorem, the β-interval in which α was contained. Let N_r be the number of α's giving to g_0, g_1, $\ldots$, g_k exactly r variations. The problem is then to determine N_r.

It will be convenient to denote the product $g_{i-1}(x)\cdot g_i(x)$ by $G_i(x)$, and the product $G_{i_1}(x)\cdot G_{i_2}(x)\ldots G_{i_s}(x)$ by $G_{i_1i_2\cdots i_s}(x)$, or briefly by $G_{(i)}(x)$. A product of g's may in like manner be written $g_{(i)}(x)$; but, for the most part, it will be convenient to write formulas in terms of the G's. It will be understood that $G(x)$ or $g(x)$, without suffix, represents unity.

There will be a permanence or a variation between $g_{i-1}(\alpha)$ and $g_i(\alpha)$ according as $G_i(\alpha)$ is positive or negative (by assumption (iii) it cannot be zero). Therefore N_r is the number of α's making exactly r of the quantities $G_1(\alpha)$, $\ldots$, $G_k(\alpha)$ negative.

Let us denote by $L_{i_1i_2\cdots}$ or $L_{(i)}$, the number of α's making the particular set

* Or let g_0, g_1, $\ldots$, g_k be any general Sturmian sequence, however obtained, in the sense that its loss of variations agrees with that of the sequence defined above.

† The reader may prefer to assume throughout that g_0 has no multiple roots in the interval, and that neither a nor b is a root of g_0.

of functions $G_{i_1}, G_{i_2}, \ldots$ negative, and the remaining G's positive. Then it is evident that N_r is equal to the sum of those L's that possess r suffixes; that is to say, $N_0 = L$, $N_1 = \sum_i L_i, \ldots, N_k = L_{12\ldots k}$.

In order to find the L's, let us connect them with another set of integers which it will be possible to determine directly. Let $M_{i_1 i_2}\ldots$, or $M_{(i)}$, be the excess of the number of α's making the product $G_{i_1 i_2}\ldots$ positive over the number making it negative. In particular, M (without suffix) is the number of roots of $f_0(x)$ in the interval. It will also be understood that $m_{(i)}$ has the same meaning with reference to $g_{(i)}$.

It is clear that a particular α contributes $+1$ or -1 to $M_{i_1 i_2}\ldots$ according as it gives negative values to an even or an odd number of the functions $G_{i_1}, G_{i_2}, \ldots$ whose suffixes appear in the symbol $M_{i_1 i_2}\ldots$. If then we denote by $\{(i), (j)\}$ the number of elements common to the two sets of integers (i) and (j), we find that the 2^k integers $M_{(j)}$ are connected with the 2^k non-negative integers $L_{(i)}$ by means of the system of 2^k equations

$$\sum_{(i)} (-1)^{\{(i),\,(j)\}} L_{(i)} = M_{(j)}, \tag{1}$$

where the summation covers all combinations of i's (including the null-set) and there is one such equation for each combination of j's.

To solve (1) for the L's we notice that, whenever (i) and (i') are not the same set,

$$\sum_{(j)} (-1)^{\{(i),\,(j)\}} (-1)^{\{(i'),\,(j)\}} = 0.$$

For, if e is some element present in one but not both of (i) and (i'), the sum on the left side is composed of pairs of equal and opposite terms differing only in the presence or absence of e in (j).

Hence, on multiplying (1) by $(-1)^{\{(i'),\,(j)\}}$ and adding, we have, after writing (i) for (i'),

$$L_{(i)} = 2^{-k} \sum_{(j)} (-1)^{\{(i),\,(j)\}} M_{(j)}. \tag{2}$$

The problem is now reduced to the determination of the M's; and for this we have the method of §2 available. Denoting by $H_{(j)}(x)$ the product $G_{(j)}(x) \cdot f_1(x)$, let us form by the Sturmian division process the sequence $S[f_0, H_{(j)}] = [f_0, H_{(j)}, H'_{(j)}, H''_{(j)}, \ldots]$. If $V_{(j)}(x)$ represents the number of variations of sign in the sequence for any given x, the required formula is, by the corollary in §2,

$$M_{(j)} = V_{(j)}(a) - V_{(j)}(b). \tag{3}$$

An alternative process for finding $M_{(j)}$ will be given in §13.

We now have a complete chain of relations leading to the values of N_r for $r = 0, 1, \ldots, k$. These relations contain the solution of the problem for two polynomials as far as the relative distribution of the α's and the intervening β's are concerned. They do not disclose the number of outlying β's at either end of the interval. These numbers, however, are easily found by observing the values of $g_0, g_1, \ldots$ when $x = a$ and b. If we introduce the fresh assumption (iv) that neither a nor b is a multiple root of g_0, and if the sequence $g_0, g_1, \ldots, g_k$ shows κ variations for $x = a$ and λ variations for $x = b$, while κ' and λ' are the highest and lowest values of r for which $N_r \neq 0$, it is clear that the numbers of marginal β's are $\kappa - \kappa'$ and $\lambda' - \lambda$, on the understanding that the interval includes b but not a.

4. The distribution function

We are interested, not in the individual numbers $L_{(i)}$, but in the sum of those among them that involve r suffixes, since this sum is equal to N_r. It will be seen that the N's are not capable of simple expression in terms of the M's. For some purposes the most convenient way of exhibiting the relation between the N's and M's is by means of a generating function

$$E(t) = \sum_{r=0}^{k} N_r t^r.$$

On writing N_r in terms of the L's and then eliminating the latter by means of (2), we find that

$$E(t) = 2^{-k} \sum_{(j)} \sum_{(i)} (-1)^{\{(i),\,(j)\}} M_{(j)} t^r, \tag{4}$$

where, in any term of the sum, r is the number of elements in (i).

Now it is not difficult to see that, for any particular set (j), s in number,

$$\sum_{(i)} (-1)^{\{(i),\,(j)\}} t^r = (1+t)^{k-s}(1-t)^s. \tag{5}$$

For the linear factors of the right member may be arranged in such order that the jth factor is $(1-t)$ for every j belonging to (j). The general term of the product (if like terms are not collected) is then the positive or negative product of a number of t's drawn from the i_1th, i_2th, ... factors, the number of negative factors in this term being $\{(i),\ (j)\}$. The complete product contains one such term for each possible combination (i), which shows that (5) is correct.

By means of (5), (4) may be reduced to the form

$$E(t) = 2^{-k} \sum_{(j)} M_{(j)} (1+t)^{k-s}(1-t)^s,$$

s being the number of elements in (j).

If P_s stands for the sum of those M's having s suffixes, we have finally the identity in t

$$\sum_{r=0}^{k} N_r t^r = E(t) = 2^{-k}\sum_{s=0}^{k} P_s(1+t)^{k-s}(1-t)^s. \tag{6}$$

The P's being known, the expansion of the right member in descending powers of t will give, by means of its coefficients, the numbers of α's in the successive β-intervals from left to right. We may call $E(t)$ the "distribution function."

Since the P's are formed from the M's in the same way as the N's from the L's, a reference to equations (1) and (2) will convince us that, corresponding to (6), there is another identity (in u)

$$\sum_{s=0}^{k} P_s u^s = \sum_{r=0}^{k} N_r(1+u)^{k-r}(1-u)^r. \tag{7}$$

In fact (7) may be obtained from (6) by the transformation $t = (1-u)/(1+u)$.

To illustrate the occasional advantage of the reciprocal formula (7) over (6), let us write the conditions that the root-distribution should be $\beta\beta\alpha\alpha\beta$, taking $k = 3$. It is required that $\sum_s P_s u^s \equiv 2(1+u)^2\,(1-u)$, so that the conditions are $P_0 = 2$, $P_1 = 2$, $P_2 = -2$, $P_3 = -2$. If a pair of polynomials does not satisfy these conditions, the fact is likely to be discovered without computing all the M's. On the other hand, the use of the t-identity (6) gives four conditions each of which involves all the eight M's.

5. Remarks on the preceding theory

(i) The relations expressed in terms of the G's in equations (1), (2), (3), and (6) may also be put in terms of the g's. Equations (1), (2), and (3) will merely exchange the capitals for small letters (to be understood in a corresponding sense), except that the factor 2^{-k} will be replaced by 2^{-k-1}; but (6) will go into a less simple form, and it is therefore convenient to keep the formulas in terms of the capital letters. In numerical work, however, the calculations will actually be made, as the following remark will show, with functions of lower degree than would appear from the general formulas.

(ii) In computing $M_{(j)}$, we have evidently the right to substitute for $G_{(j)}$ or $H_{(j)}$ any polynomial assuming the same signs for all the α's. For example, $M_{123} = m_{03}$; hence H_{123} may be replaced by $h_{03} = g_0 g_3 f_1$.

We may also subtract from any $G_{(j)}$ or $H_{(j)}$ the product of any polynomial by f_0, and the residue so obtained may be used in place of the original function, not only for the calculation of $M_{(j)}$, but also in the formation of higher products.

(iii) When some $M_{(i)}$ is equal to $\pm M$, every $M_{(j)}$ calculated will at the same time furnish the value of some other. For, if (i') and (i'') together make up (i), and if (j) is any set not overlapping (i), we know by (ii) that $M_{(i',j)} = \pm M_{(i'',j)}$, the ambiguity being in agreement with the former, where (i, j) means the set of elements belonging to either (i) or (j).

In particular, if it is known that $M_i = \pm M$ for every i in some set (e), it is no longer necessary to calculate independently any $M_{(j)}$ whose set of suffixes has elements in common with (e).

These ideas will be generalized in §10.

(iv) In certain cases, as when g_0 is known to have all its roots real and distinct, g_0 and its successive derivatives will serve as the sequence $(g_0, g_1, \ldots, g_k)$. In any case the Budan-Fourier theorem shows that the root-distribution indicated by the use of the derivatives will differ from the true distribution at most by the presence of pairs of consecutive β's. It will be exact if no such pairs are indicated.

(v) Failure of any of the hypotheses (i), (ii), or (iii) of §3 will always lead to reducibility of f_0, if the degree of f_0 is not less than that of g_0.

6. Numerical examples

In the following examples, substitutions in accordance with §5 (ii) are freely made without a corresponding change of notation.

(i) Let $f_0(x) = x^3 + 3x^2 - 3x - 1, \quad g_0(x) = x^2 - 4x - 7.$

Then $f_1 = x^2 + 2x - 1;\ g_1 = x - 2,\ g_2 = 1$. Therefore $G_1 = -9x^2 + 4x + 15, G_2 = x - 2, G_{12} = g_0 = x^2 - 4x - 7$. Hence

$$\begin{array}{llll}
H = x^2+2x-1, & H_1 = -17x^2 + 28x - 1, & H_2 = -3x^2 - 2x + 3, & H_{12} = x^2 - 12x + 1,\\
H' = x, & H'_1 = -83x + 23, & H'_2 = 8x - 3, & H'_{12} = -11x + 1\\
H'' = 1, & H''_1 = -1, & H''_2 = -1, & H''_{12} = 1,
\end{array}$$

$$\begin{array}{llll}
V(-\infty) = 3, & V_1(-\infty) = 2, & V_2(-\infty) = 0, & V_{12}(-\infty) = 1,\\
V(0) = 1, & V_1(0) = 2, & V_2(0) = 2, & V_{12}(0) = 1,\\
V(\infty) = 0, & V_1(\infty) = 1, & V_2(\infty) = 3, & V_{12}(\infty) = 2.
\end{array}$$

If the interval is $(-\infty, \infty)$,

$$M = 3, \qquad M_1 = 1, \qquad M_2 = -3, \qquad M_{12} = -1.$$

Hence $E(t) = \frac{1}{4}\{3(1+t)^2 + (1+t)(1-t) - 3(1+t)(1-t) - (1-t)^2\} = t^2 + 2t$. Also $\kappa = 2$ and $\lambda = 0$; so that the arrangement is $\alpha\beta\alpha\alpha\beta$.

If the interval is $(-\infty, 0)$,

$$M = 2, \qquad M_1 = 0, \qquad M_2 = -2, \qquad M_{12} = 0.$$

Therefore $E(t) = \frac{1}{4}\{2(1+t)^2 - 2(1-t^2)\} = t^2 + t$. Moreover, λ is now 1, showing that there are no β's on the upper side, and the arrangement is $\alpha\beta\alpha$.

In both cases M_{12} might have been found as the negative of M_1, as explained in §5 (iii).

(ii) Let $f_0 = x^2 - 3x - 1$, $g_0 = x^3 - 3x^2 - 3x - 1$; $a = -\infty$, $b = \infty$.

We find that $M = 2$, $M_1 = 0$, $M_2 = 0$, $M_{12} = -2$, $M_3 = -2$; and it follows that $M_{13} = -M_1 = 0$, $M_{23} = -M_2 = 0$, $M_{123} = -M_3 = 2$.

Hence $E(t) = \frac{1}{8}\{2(1+t)^3 - 2(1+t)^2(1-t) - 2(1+t)(1-t)^2 + 2(1-t)^3\} = 2t^2$. Since $\kappa = 2$ and $\lambda = 1$, the order of roots is $\alpha\alpha\beta$.

7. More than two polynomials

It is always possible to find the relative arrangement of roots for a system of polynomials by applying the method of §§3 and 4 to the several pairs. It may, however, be interesting to exhibit the solution of the problem by means of a single formula analogous to (6).

Let one of the polynomials be $f_0(x)$, having roots $\alpha_1, \alpha_2, \ldots$ in (a, b); and let the other polynomials (in finite number) be $g'_0, g''_0, \ldots$. Let all the preceding notation be applied, with accents wherever necessary; and further let the product $G'_{(i')}G''_{(i'')} \ldots$ be abbreviated to $G_{[(i)]}$.*

It will be assumed that neither a nor b is a root of f_0, and that f_0 has in (a, b) no multiple root or root in common with any of the Sturmian functions g'_0, $g'_1, g'_2, \ldots, g'_{k'}, g''_0, \ldots, \ldots$.

Let $N_{r'r''}\ldots$ or $N_{[r]}$ be the number of α's lying in that β-interval in which the sequence $g'_0, \ldots, g'_{k'}$ has r' variations, $g''_0, \ldots, g''_{k''}$ has r'' variations, and so on. By reasoning similar to that of §§3 and 4, we find that $N_{[r]}$ is the coefficient of $t'^{r'}t''^{r''}\ldots$ in the generating function

$$(8)\quad E(t', t'', \ldots) = 2^{-k'-k''-\cdots}\sum_{[(j)]} M_{[(j)]}(1+t')^{k'-s'}(1-t')^{s'}(1+t)^{k''-s''}(1-t'')^{s''}\ldots,$$

where s' is the number of elements in (j'), s'' the number in (j''), etc.

It is evident that the remarks made in §5 apply without essential modification to the present case.

* It will be noted that [ ] is the symbol of a permutation, ( ) of a combination.

8. Generalization of the range

Hitherto we have taken account of all the roots of f_0 that lie in a given real interval (a, b). Let us now consider how formula (8) or (6) must be modified when only those roots are counted which, besides lying in (a, b), satisfy a given system of algebraic inequalities.

Let us have then, in addition to the functions already assumed, a system of n auxiliary polynomials $G^1(x), G^2(x), \ldots, G^n(x)$ without roots in common with f_0, and let the inequalities in question be

$G^i < 0$ for every i in a certain set (e);
$G^i > 0$ for the remaining i's.

Let the α's satisfying these inequalities be denoted generally by $\bar{\alpha}$.

Let $G^{(i)}_{[(j)]}$ denote the product of all the G's whose indices (upper or lower) appear in the symbol, and let us define in connection with this product the corresponding quantities of types M, H, and V. Our object being to find $\overline{N}_{[r]}$, the number of $\bar{\alpha}$'s in a particular β-interval, we shall study the generating function $\overline{E}(t', t'', \ldots) = \sum_{[r]} \overline{N}_{[r]} t'^{r'} t''^{r''} \ldots$.

Let us denote by $E^{(i)}(t', t'', \ldots)$ the function

$$2^{-k'-k''-\cdots} \sum_{[(j)]} M^{(i)}_{[(j)]} (1 + t')^{k'-s'} (1 - t')^{s'} \ldots,$$

obtained from $E(t', t'', \ldots)$ by affixing the set of upper indices (i) to every M, observing that $E^{(i)}$ is not in general the generating function of a distribution and that some of its coefficients may be negative. It will be called a "modified distribution function." There is clearly a set of relations, analogous with (1),

$$\sum_{(e)} (-1)^{\{(e),\,(i)\}} \overline{E} = E^{(i)}, \tag{9}$$

where the summation is for all combinations (e) out of $1, 2, \ldots, n$, and where the $\overline{E}$ in any term is formed according to the inequalities associated with the (e) in question. From these we obtain, as (2) was obtained,

$$\overline{E} = 2^{-n} \sum_{(i)} (-1)^{\{(e),\,(i)\}} E^{(i)}. \tag{10}$$

The M's appearing in $E^{(i)}$ being given by the formula

$$M^{(i)}_{[(j)]} = V^{(i)}_{[(j)]}(a) - V^{(i)}_{[(j)]}(b), \tag{11}$$

the function $\overline{E}$ is determined.

In practice we can of course arrange to have all inequalities of the form $G^i > 0$, when (10) reduces to $\bar{E} = 2^{-n}\sum_{(i)}E^{(i)}$.

There are two ways in which, starting from the range $(-\infty, \infty)$, we may arrive at the interval (a, b) by the use of (10). (i) We may use the two inequalities $G^1 = x - a > 0$, $G^2 = x - b < 0$, obtaining $\bar{E} = \frac{1}{4}(E + E^1 - E^2 - E^{12})$. (ii) We may use the single inequality $G^1 = (x - a)(x - b) < 0$, obtaining $\bar{E} = \frac{1}{2}(E - E^1)$. Of these, (ii) still gives correct results when $a \geqq b$, if we agree that (a, b) and (b, a) are the same interval; while (i) is correct in this case if by (a, b) we mean the interval $a < x < b$. In the method of §3 it is necessary that $a \leqq b$.

9. The modified distribution function

It is interesting to examine a little the meaning of the modified function $E^{(i)}$, at least in a simple case. Without being a true distribution function, it has some relation to the distribution, as equation (9) shows.

Let us consider the case of two polynomials f_0 and g_0 with a single auxiliary function G^1, and denote now by E^+ the generating function of the root-distribution in that part of (a, b) where G^1 is positive, and by E^- the same for G^1 negative. With E^1 as already defined (as a special case of $E^{(i)}$), we have, by (9), $E = E^+ + E^-$ and $E^1 = E^+ - E^-$; so that $E - 2E^- = E^1 = -E + 2E^+$.

It follows that the general coefficient N_r^1 in E^1 does not exceed in absolute value the corresponding N_r, and differs from it by an even number. In particular, $|N_r^1|$ will correctly enumerate the α's in the β-interval whenever either N_r^+ or N_r^- is zero; that is, in any β-interval where the α's are not separated by a root of G^1.

The information given by E^1 is complete when $\sum_r|N_r^1| = M$. In other cases the total error in the enumeration of the α-groups by E^1 is $M - \sum_r|N_r^1|$.

Two special cases may be noted.

(i) If $G^1 = g_0$, all the α-groups will remain undivided, and E^1 will give exact information. In fact, for this case, if g_0 has no roots in (a, b) of even multiplicity, $E^1(t) = \pm E(-t)$.

(ii) On taking $G^1 = g_0f_1$, we find that the function $H_{(j)}$ is replaced by $g_0G_{(j)}$. Every N_r^1 is now 0 or ± 1, according as N_r is even or odd. The α-groups have therefore been reduced modulo 2. To arrive at the Cauchy-Sylvester theory, we must reduce both α- and β-groups, which is accomplished by making $t = 1$. In fact, $E^1(1)$ is equal to M^1, which is the loss of variations in the sequence $S(f_0, g_0)$; in other words, the Cauchy index.

10. The M-inequalities

Although the $M_{(j)}$ are independent quantities, in the sense that they are connected by no identical equation, they satisfy a number of linear inequalities, which, in many cases, enable us to dispense with a part of the work in computing them. These inequalities express the fact that the $L_{(i)}$ are essentially $\geqq 0$, so that they take the form

$$\sum_{(j)}(-1)^{[(i),(j)]}M_{(j)} \geqq 0. \tag{12}$$

On account of these relations, a single equation in the M's may be equivalent to a number of equations. For example, the equation $\sum_s P_s(k-2s) = 0$, being the same as $\sum_i L_i = 0$, breaks up into the k equations $L_i = 0$ ($i = 1, \ldots, k$), which again may be written in terms of the M's.

For the purpose of direct numerical application of the theory, the most important consequences of (12) are contained in the following theorem.

If, for some set (e) contained in a second given set (d),

$$\sum_{(i)\text{ in }(d)}(-1)^{[(i),(e)]}M_{(i)} = 0, \tag{A}$$

it will follow that, for every (j) without elements in common with (d),

$$\sum_{(i)\text{ in }(d)}(-1)^{[(i),(e)]}M_{(i,j)} = 0, \tag{B}$$

where (i, j) is the set whose elements are those of (i) and those of (j). The summation is for all subsets (i) of (d). The proof is that (A) expresses the vanishing of the sum of those $L_{(e,r)}$ for which (r) has no elements in common with (d). These $L_{(e,r)}$ must separately vanish. Hence

$$\sum_{(r)}(-1)^{[(r),(j)]}L_{(e,r)} = 0,$$

which reduces to (B).

It is natural to compute the M's in some such order as M, M_i, M_j, M_{ij}, M_p, M_{ip}, M_{jp}, M_{ijp}, $\ldots$. If some $M_{(d)}$ is found, with preceding M's,to satisfy a relation of type (A), all the M's whose suffixes include (d) are given by equations of type (B) in terms of M's already found. The smaller the set (d), the more numerous are the consequences. When (d) has only one element, we get the rule of §5 (iii) with respect to the condition $M_i = \pm M$, and the values of one half of the M's are given in terms of the others. If there are two elements in (d), (A) will have one of three possible forms; for instance, if (d) is 1, 3, and (e) is 3, (A) becomes $M + M_1 - M_3 - M_{13} = 0$, and (B) is

of the form $M_{(p)} + M_{1(p)} - M_{3(p)} - M_{13(p)} = 0$, which gives a quarter of the M's, namely those of the form $M_{13(p)}$.

This theorem remains true when (j) is *any* set, and also when the individual suffixes are replaced by sets of suffixes, provided we understand by (i, j) the set of all elements occurring in one but not both of (i) and (j). Thus, from $M + M_{12} + M_{13} + M_{23} = 0$ we deduce $M_{14} + M_{24} + M_{34} + M_{1234} = 0$.

A reason for the frequent intervention of this principle in shortening numerical work is in the limited choice of values for $M_{(d)}$, which narrows as the number of suffixes increases. Since $\sum_{(i) \text{ in } (d)} M_{(i)}$ must be divisible by 2^q, where q is the number of elements in (d), $M_{(d)}$ is known beforehand modulo 2^q, and this alone will enable us to foretell the value of $M_{(d)}$ if $2^q > 2M$.

There is no difficulty in extending the principle of this section to the more general methods of §§7 and 8.

11. Application to special problems

In the following applications (i) to (vi) the reference is to two polynomials in a single interval (finite or infinite).

(i) The question whether all the α's are to be found in a particular set of β-intervals reduces to the question of the linear dependence of $\sum_s P_s u^s$ on $(1+u)^{k-r_1}(1-u)^{r_1}$, $(1+u)^{k-r_2}(1-u)^{r_2}$, $\ldots$; but in particular cases, the conditions may take various forms through the principle discussed in §10.

Thus (under assumption (iv) of §3) the β's will be entirely unseparated by the α's if and only if the sum vanishes of all L's with more than λ and less than κ suffixes, which reduces to a single equation in the M's.

In particular, if all the roots of g_0 are simple and interior to (a, b), so that $\kappa = k$ and $\lambda = 0$, a necessary and sufficient condition that the α's do not separate them is that $M - L - L_{12\cdots k} = 0$; which reduces, first to $\sum_{(i) \text{ even}} [M - M_{(i)}] = 0$, and finally to the condition that all $M_{(i)}$ with even numbers of suffixes are equal. By §5 (iii), it is necessary (though not sufficient) that the $M_{(i)}$ with odd numbers of suffixes are also equal. When the roots are so arranged, the manner of partition is completely determined by $N_0 = \frac{1}{2}(M + M_1)$, $N_k = \frac{1}{2}(M - M_1)$.

(ii) A condition for k β's, each less than any one of the α's, is $M_i = M$ for every i.

For k β's, each greater than any α, $M_i = -M$ for every i.*

(iii) For the α's to occur in alternate β-intervals only, $P_s = \epsilon P_{k-s}$ for every s, where ϵ is either 1 or -1; which, by §5 (iii), is equivalent to the one con-

* See also the note to (vi).

dition $M = \pm m_{0k}$. This condition may also be obtained from elementary considerations.

(iv) The partition of the α's is ordinally symmetric about the position where the g_i's have $k/2$ variations if and only if $P_r = 0$ for every odd r.

(v) To have an arrangement consisting of $(n+1)$ groups of r α's each, alternating with n groups of s β's each, where $ns = k$, it is necessary and sufficient that

$$r[(1+u)^{ns} + (1+u)^{(n-1)s}(1-u)^s + \cdots + (1-u)^{ns}] = P_0 + P_1 u + \cdots + P_k u^k.$$

It is therefore necessary that $P_i = 0$ for every odd i.

If s is even, the equations of condition reduce to one more than half their original number, owing to the inclusion of the equation $M = M_{12\ldots k}$, and involve only one more than half the M's. For instance, (taking $n = r = s = 2$) the roots of a sextic and a quartic occur alternately in pairs whenever $M = M_{1234} = 6$, $M_{12} + M_{13} + M_{23} = 10$, and $M_1 + M_2 + M_3 + M_4 = 0$.

When $r = s = 1$, the conditions are $P_{2i} = \binom{k+1}{2i+1}$, $P_{2i+1} = 0$; but the following method is simpler.

(vi) Let g_0 have no multiple roots in (a, b). The modified process with the auxiliary function $G^1 = g_0 f_1$ shows that a necessary and sufficient condition for k (or $k+1$) α's alternating with k β's is $M = k$ (or $k+1$), $M^1 = \pm M$. These are Sylvester's results, and represent the case where his restricted theory gives complete information. The tests involve very little calculation. It is easy to see that k may here be interpreted as the number of β's*.

(vii) The theory of the generalized range in §8 enables us to find rationally the number of separate intervals of the real number system satisfying a set of algebraic inequalities

$$(A) \qquad G^i > 0 \qquad (i = 1, 2, \ldots, n),$$

the G^i being real polynomials in x without common or multiple roots in the range $(-\infty, \infty)$.

To complete the system of §8, let f_0 denote the first derivative of the product $G^1G^2\ldots G^n$, and suppose that f_0 has no real multiple root. Let $g_0 = 1$. Since $k = 0$, the E-functions reduce to constants (the corresponding M's). Let us introduce another auxiliary function $G^0 = f_1$, and let M^+ denote the number of real roots of f_0 making G^0 positive and satisfying (A), M^- the number making G^0 negative and satisfying (A).

* The result of (vi) may be used to prove the following theorem. Let f_i and g_i be the ith derivatives of f_0 and g_0, and let n_i be the loss of variations between a and b in the sequence $S(f_{i-1}g_i, f_i g_{i-1})$. Let p_i be the degree of $f_{i-1}g_i$, and q the degree of the lower (or both) of f_0 and g_0. A necessary and sufficient condition that the roots of f_0 and g_0 may be all real and simple and in (a, b), with the least root of f_0 greater than the greatest root of g_0, is that $n_i = p_i$ $(i = 1, \ldots, q)$.

Equation (10) degenerates in the two respective cases to

$$M^+ = 2^{-n-1}\sum_{(i)}\{M^{(i)} + M^{0(i)}\}, \quad M^- = 2^{-n-1}\sum_{(i)}\{M^{(i)} - M^{0(i)}\},$$

(i) denoting in turn every subset of $1, 2, \ldots, n$. A little consideration then shows that the number of separate finite intervals is $M^- - M^+$, that is $-2^{-n}\sum_{(i)}M^{0(i)}$. The question of infinite regions may be settled by means of the signs of the coefficients of the highest terms of the G's.

It will be seen that, as a result of adjoining the derivative of f_0, we may remove a square factor from $H^{0(i)}$, so that $M^{0(i)}$ is the number of variations lost between $-\infty$ and ∞ by the sequence $S(f_0, G^{(i)})$, and G^0 does not actually enter into the calculations.

As an example, let $G^1 = 1 - x^2$, $G^2 = x^2 - 2x$ $(n = 2)$. Then $f_0 = -2x^3 + 3x^2 + x - 1$; M^0 = loss of variations in $S(f_0, 1) = 0 - 1 = -1$; M^{01} = loss of variations in $S(f_0, G^1) = 2 - 1 = 1$; $M^{02} = -1$; $M^{012} = -3$. Hence the number of finite intervals is $-\frac{1}{4}(-1 + 1 - 1 - 3) = 1$. There is no infinite region.

(viii) A second method for the problem of (vii) involves generally more sequences, but simpler ones, and allows f_0 to have real multiple roots. Let ${}_iM^{(j)}$ be the value of $M^{(j)}$ when G^i is used in the place of f_0, (j) not containing i. Let I be the number of infinite regions satisfying (A), as found from the signs of the highest terms. It may be shown then that the number of separate regions, finite and infinite, defined by (A) is $2^{-n}\sum_i \sum_{(j)} {}_iM^{(j)} + \frac{I}{2}$.

Thus, if $G^1 = x^2 - 6x + 8$, $G^2 = x^2 + 6x + 8$, and $G^3 = x$ $(n = 3)$, we shall find that ${}_1M = {}_1M^2 = {}_1M^3 = {}_1M^{23} = 2$, ${}_2M = {}_2M^1 = 2$, ${}_2M^3 = {}_2M^{13} = -2$, ${}_3M = {}_3M^1 = {}_3M^2 = {}_3M^{12} = 1$; and the functions are all positive to the right, but not to the left. Hence the number of regions is $\frac{1}{8}(6\times 2 - 2\times 2 + 4\times 1) + \frac{1}{2} = 2$.

12. Restricted distributions

It will sometimes happen that, before applying the general theory, we have a partial knowledge of the root-distribution which enables us to assume for E an expression of the type $e + \sum_{i=1}^{n} c_ie_i$, where the coefficients c_i are independent of t (or of $t', t'', \ldots$) and fewer than $k + 1$ in number, and e, e_i are known functions of t(or of $t', t'', \ldots$). In such a case it is not usually necessary to use all the M's in determining the distribution. If, starting with the equation

$$e + \sum_i c_ie_i = 2^{-k}\sum_s P_s(1 + t)^{k-s}(1 - t)^s,$$

we differentiate successively, and then put t in turn equal to 1 and -1, we get $2k + 2$ equations which, if the e_i are linearly independent, determine the n

constants c_i. From these equations we may select any n whose determinant is not zero, and we shall naturally do so in such a way as to use as few M's as possible, noting that, when $t = 1$, the rth derivative of E involves no M with more than r suffixes, and that, when $t = -1$, it involves none with fewer than $k - r$.

Let $\Delta^{\rho_1 \dots \rho_\nu}_{\sigma_1 \dots \sigma_{n-\nu}}$ be the n-rowed determinant whose first ν rows consist of the derivatives of order $\rho_1, \rho_2, \dots, \rho_\nu$ of the e_i, with $t = 1$, and whose last $n - \nu$ rows consist of derivatives of order $\sigma_1, \sigma_2, \dots, \sigma_{n-\nu}$, with $t = -1$. Let us examine this determinant in some special cases. The problems most naturally presenting themselves are those in which e_i is of the form $t^{r_i}(1 + t)^\sigma (1 - t)^\rho \varphi(t)$ $(i = 1, \dots, n)$, where $\varphi(t)$ is not divisible by $1 + t$ or $1 - t$. Let μ of the r_i be even and $n - \mu$ of them odd. It may then be shown that $\Delta^{\rho, \rho+1, \dots, \rho+\nu-1}_{\sigma, \sigma+1, \dots, \sigma+n-\nu-1}$ differs from zero when $\nu = 0$ or n, as also when the smaller of ν and $n - \nu$ is equal to the smaller of μ and $n - \mu$, but is equal to zero when the smaller of ν and $n - \nu$ is greater than the smaller of μ and $n - \mu$.

For example, whenever $\kappa - \lambda$ is less than k, we have an instance with $\rho = \sigma = 0$ and $\varphi(t) = 1$. If then g_0 has only two roots in (a, b), the positions of the α's with respect to them may be found by means of P_0, P_1, and P_2; P_k, P_{k-1}, P_{k-2}; P_0, P_1, P_k; or P_0, P_k, P_{k-1}.

Another example is afforded by the case where the distributions of the α's and the β's with respect to some special set of numbers are already known, so that their mutual arrangement is known subject to certain transpositions of adjacent roots. In such a case $\rho = 1$, $\sigma = 0$, and $\varphi(t) = 1$. Thus, the arrangement being $(\alpha\beta)(\alpha\alpha\beta\beta)\beta(\alpha\alpha\beta)$, with the order in each bracketed group undetermined, we have a case where μ and $n - \mu$ are 1 and 3 (or 3 and 1), and the complete determination is made by means of P_1, P_k, P_{k-1}, P_{k-2}, but not by means of P_1, P_2, P_k, P_{k-1}.

A second type of restricted distribution has its generating function of the form $e + \varphi(t) \cdot \sum_{i=1}^{n} c_i t^{r_i}$, where now only the exponents r_i are unknown. If $\varphi(1)$ is not zero, we find, by successive differentiation and putting $t = 1$, $\sum_i c_i r_i^j$ in terms of $P_0, P_1, \dots, P_j$ for $j = 1, 2, \dots$. The first n of the resulting equations must, if the c_i are all different from zero, be solvable rationally in the n unknown r_i, though possibly not without ambiguity. Thus, if the first two equations so obtained are $2r_1 + r_2 = 12$, $2r_1^2 + r_2^2 = 54$, either $r_1 = 3$ and $r_2 = 6$, or $r_1 = 5$ and $r_2 = 2$. The value of $2r_1^3 + r_2^3$ will decide.

Generally, it may be proved that, if the first $2n$ equations are used, there cannot be two solutions in non-negative r's unless they are mere permutations of one another determining the same distribution.*

When the coefficients c_i are all positive, the n equations $\sum_{i=1}^{n} c_i r_i^j = C_j$

* This can be proved from Descartes' rule by making the transformation $j = \log y$, $\log r_i = p_i$.

$(j = 1, 2, \ldots, n)$ do not admit more than one solution satisfying the conditions $r_1 \geqq r_2 \geqq \cdots \geqq r_n \geqq 0$.* Thus, in the example already used, if the α's are known to fall into a group of two to the left and a single one to the right, the fact that these are in the intervals of five and two variations (three and six being now excluded) will be discovered with the aid of P_0, P_1, and P_2 only.

In particular, if the c_i are all unity, the first n equations obtained will always give a unique solution of the problem. Clearly the c's may always be taken as unity, by using enough of them.

There are two conspicuous cases to which this method applies. The first is that in which $M < k + 1$. We may then take $E(t) = \sum_{i=1}^{M} t^{r_i}$, and the values of $P_0, P_1, \ldots, P_M$ will suffice.

The other case is that of the relative distribution of the roots of a polynomial and those of its derivative over the range $(-\infty, \infty)$. If $g_0 = f_1$, E is of the form $\sum_{i=\lambda}^{\kappa} t^i - (1+t)\sum_{i=1}^{n} t^{r_i}$; and the present method gives $\kappa - \lambda + 1 - 2n = P_0$, $\sum_{i=\lambda}^{\kappa} i - 2\sum_i r_i - n = kP_0/2 - P_1/2$, etc. The first equation gives n, and the next n equations give in succession $\sum_i r_i, \sum_i r_i^2, \ldots, \sum_i r_i^n$ in terms of P_0, $P_1, \ldots, P_n$, thus determining the positions of the $\kappa - \lambda$ turning-points of f_0 with respect to the roots.

If in this problem the range is not $(-\infty, \infty)$ but (a, b), the formula for $E(t)$ is $\sum_{i=\lambda}^{\kappa} t^i - (1+t)\sum_{i=1}^{n} t^{r_i} - \frac{1}{2}\{1 - \operatorname{sgn}[f_0(b)f_1(b)]\}t^\lambda - \frac{1}{2}\{1 + \operatorname{sgn}[f_0(a)f_1(a)]\}t^\kappa$, the theory being otherwise the same.

13. Method by quadratic forms

An alternative method for the determination of $M_{[(j)]}^{(i)}$ depends on the theory of real quadratic forms, and corresponds to Hermite's theorem on a single polynomial, as the previous method corresponds to that of Sturm.

Let $\xi_1, \xi_2, \ldots, \xi_p$ be all the roots of f_0, real and imaginary, counted each as often as its multiplicity requires. We shall suppose still that assumptions (i), (ii), and (iii) of §3 hold good.

By §8, it will be sufficient to consider the case where the range is $(-\infty, \infty)$, and for this range we know, by Hermite's principle,† that $M_{[(j)]}^{(i)}$ is the excess of positive over negative coefficients in any sum of real square terms to which a real linear transformation on the T's will reduce the quadratic form

$$\sum_{r=1}^{p}\left(G_{[(j)]}^{(i)}(\xi_r)\cdot\left\{\sum_{s=0}^{p-1}\xi_r^{\,s}T_s\right\}^2\right).$$

* This follows in a similar way from a theorem by Laguerre. See **Acta Mathematica**, vol. 4 (1884), pp. 111–114.

† *Sur le nombre des racines d'une équation algébrique comprises entre des limites données*, Extrait d'une lettre à M. Borchardt, **Journal für Mathematik**, vol. 52 (1856), p. 39.

The matrix of this form is $\left(\sum_{r=1}^{p} \xi_r^{s+t} G_{[(j)]}^{(i)}(\xi_r)\right)_{\substack{s=0,\cdots,p-1 \\ t=0,\ldots,p-1}}$.

If any sequence of principal minors of this matrix is selected (as it may always be) such that each minor contains the preceding and no two vanish in succession, and if $W_{[(j)]}^{(i)}$ is the number of variations of sign in the sequence, we have the formula

$$M_{[(j)]}^{(i)} = p - 2\, W_{[(j)]}^{(i)}. \tag{13}$$

In numerical work by this method, a large part of the labor will occur in evaluating the determinants. As to the determination of the elements of the matrix, the theory of partial fractions provides a convenient method. For the element in the $(s+1)$th row and $(t+1)$th column is the coefficient of x^{-s-t-1} in the expansion of $H_{[(j)]}^{(i)}(x)/f_0(x)$ in descending powers of x. Since $H_{[(j)]}^{(i)}$ is the product of f_1 by a number of G's, all the necessary expansions may be obtained by multiplications from the single expansion representing $f_1(x)/f_0(x)$.

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A GENERAL THEORY OF CONJUGATE NETS*

BY

ERNEST P. LANE

1. Introduction

In the present paper we present a new method for studying conjugate nets, which possesses many advantages over those employed hitherto. We first refer the sustaining surface to its asymptotic net. We then find, by referring the surface to any one of its conjugate systems as a parametric net, that all of the projective properties of this net are expressible in terms of those quantities which determine the sustaining surface and one other function, which may be chosen arbitrarily, and which then determines the most general conjugate net on the surface. Thus one can tell at a glance which properties of a conjugate net are really peculiar to the net, and which others are due to the character of the sustaining surface.

This paper contains, as applications of the method, besides some other things, the demonstrations of a number of new theorems recently discovered by Wilczynski, and discussed by him orally at the meeting of the Society at Chicago in December, 1920. He has withdrawn his own proofs in favor of those here presented on account of the great simplification accomplished thereby.†

The method which is developed in this paper was suggested by one of G. M. Green's memoirs, entitled a *Memoir on the general theory of surfaces and rectilinear congruences*.‡ We have in fact preserved the notation which Green used in section 16 of his paper, concerning *General theorems on conjugate nets*.

2. A surface referred to its asymptotic net

Let

$$y^{(k)} = y^{(k)}(u, v) \qquad (k = 1, 2, 3, 4), \tag{1}$$

be the homogeneous coördinates of a point P_y in projective space of three dimensions. As the variables u and v vary over their ranges, P_y describes a sur-

* Presented to the Society, March 26, 1921.

† These theorems will be quoted in the following pages as Wilczynski, *Oral Communication*, Dec., 1920.

‡ These Transactions, vol. 20 (1919), pp. 79–153. Cited as *Congruences*.

face S_y. Let us assume that S_y is non-developable and does not degenerate into a single curve, and let the curves $u =$ const. and $v =$ const. be the asymptotic lines on S_y. Then the four functions $y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}$ constitute a fundamental set of linearly independent solutions of a completely integrable system of partial differential equations, which may be reduced to the form*

$$(2) \qquad y_{uu} + 2by_v + fy = 0, \qquad y_{vv} + 2a'y_u + gy = 0.$$

Conversely, every completely integrable system of form (2) defines, except for a projective transformation, a non-degenerate non-developable surface referred to its asymptotic lines.

The coefficients of system (2) are connected by the conditions of complete integrability

$$(3) \qquad \begin{aligned} a'_{uu} + g_u + 2ba'_v + 4a'b_v &= 0, \\ b_{vv} + f_v + 2a'b_u + 4ba'_u &= 0, \\ g_{uu} + 4gb_v + 2bg_v &= f_{vv} + 4fa'_u + 2a'f_u. \end{aligned}$$

The form (2) is not unique, but is preserved under all transformations of the type

$$(4) \qquad \bar{u} = U(u), \quad \bar{v} = V(v), \quad \bar{y} = C\sqrt{U'V'}\, y.$$

In order that any differential equation of the form $\alpha dv^2 + \beta dvdu + \gamma du^2 = 0$ may determine a conjugate net on S_y we must have $\beta = 0$. Excluding the limiting case when a conjugate net degenerates into one of the families of asymptotic lines, we have $\alpha\gamma \neq 0$, and may write $\alpha = 1, \gamma = -\lambda^2$. Therefore *any conjugate net on S_y may be defined by a differential equation of the form*

$$(5) \qquad dv^2 - \lambda^2 du^2 = 0,$$

where λ is a function of u and v which is nowhere zero in the region under consideration, but which is subject to no other restriction.

3. Transformation of curvilinear coördinates

We shall now make a transformation of curvilinear coördinates so that the arbitrary conjugate net (5) may become the parametric net for the surface S_y defined by (2). To this end let us consider a proper transformation of the form

$$(6) \qquad \bar{u} = \varphi(u, v), \quad \bar{v} = \psi(u, v).$$

* Wilczynski, *Projective differential geometry of curved surfaces* (First Memoir), these Transactions, vol. 8 (1907), p. 233.

We have the following well known formulas of differentiation for any function $y(u, v)$:

$$
\begin{aligned}
y_u &= \varphi_u y_{\bar{u}} + \psi_u y_{\bar{v}}, \qquad y_v = \varphi_v y_{\bar{u}} + \psi_v y_{\bar{v}}, \\
y_{uu} &= \varphi_u^2 y_{\bar{u}\bar{u}} + 2\varphi_u \psi_u y_{\bar{u}\bar{v}} + \psi_u^2 y_{\bar{v}\bar{v}} + \varphi_{uu} y_{\bar{u}} + \psi_{uu} y_{\bar{v}}, \\
y_{vv} &= \varphi_v^2 y_{\bar{u}\bar{u}} + 2\varphi_v \psi_v y_{\bar{u}\bar{v}} + \psi_v^2 y_{\bar{v}\bar{v}}\ \varphi_{vv} y_{\bar{u}} + \psi_{vv} y_{\bar{v}}, \\
y_{uv} &= \varphi_u \varphi_v y_{\bar{u}\bar{u}} + (\varphi_u \psi_v + \psi_u \varphi_v) y_{\bar{u}\bar{v}} + \psi_u \psi_v y_{\bar{v}\bar{v}} + \varphi_{uv} y_{\bar{u}} + \psi_{uv} y_{\bar{v}}.
\end{aligned} \tag{7}
$$

Let us suppose that the functions φ and ψ satisfy the partial differential equations

$$\varphi_u = -\lambda \varphi_v, \qquad \psi_u = \lambda \psi_v. \tag{8}$$

Differentiation of $\bar{u}$ and $\bar{v}$ in (6) gives

$$d\bar{u} = (dv - \lambda du)\varphi_v, \qquad d\bar{v} = (dv + \lambda du)\psi_v,$$

so that the curves $\bar{u} =$ const. and $\bar{v} =$ const. will form two component families of the conjugate net (5).

Using (8), we may express all of the u-derivatives of φ and ψ as functions of λ and derivatives of φ and ψ with respect to v only. The second order derivatives involving u are thus expressed by the relations

$$
\begin{aligned}
\varphi_{uv} &= -\lambda_v \varphi_v - \lambda \varphi_{vv}, \quad \varphi_{uu} = (\lambda \lambda_v - \lambda_u)\varphi_v + \lambda^2 \varphi_{vv}, \\
\psi_{uv} &= \lambda_v \psi_v + \lambda \psi_{vv}, \quad \psi_{uu} = (\lambda \lambda_v + \lambda_u)\psi_v + \lambda^2 \psi_{vv}.
\end{aligned} \tag{9}
$$

We are able therefore to eliminate all of the u-derivatives of φ and ψ from equations (7). In this way we obtain

$$
\begin{aligned}
y_u &= -\lambda \varphi_v y_{\bar{u}} + \lambda \psi_v y_{\bar{v}}, \qquad y_v = \varphi_v y_{\bar{u}} + \psi_v y_{\bar{v}}, \\
y_{uu} &= \lambda^2 \varphi_v^2 y_{\bar{u}\bar{u}} - 2\lambda^2 \varphi_v \psi_v y_{\bar{u}\bar{v}} + \lambda^2 \psi_v^2 y_{\bar{v}\bar{v}} + [(\lambda\lambda_v - \lambda_u)\varphi_v + \lambda^2 \varphi_{vv}] y_{\bar{u}} \\
&\qquad + [(\lambda\lambda_v + \lambda_u)\psi_v + \lambda^2 \psi_{vv}] y_{\bar{v}}, \\
y_{vv} &= \varphi_v^2 y_{\bar{u}\bar{u}} + 2\varphi_v \psi_v y_{\bar{u}\bar{v}} + \psi_v^2 y_{\bar{v}\bar{v}} + \varphi_{vv} y_{\bar{u}} + \psi_{vv} y_{\bar{v}}, \\
y_{uv} &= -\lambda \varphi_v^2 y_{\bar{u}\bar{u}} + \lambda \psi_v^2 y_{\bar{v}\bar{v}} - (\lambda_v \varphi_v + \lambda \varphi_{vv}) y_{\bar{u}} + (\lambda_v \psi_v + \lambda \psi_{vv}) y_{\bar{v}}.
\end{aligned} \tag{10}
$$

Solving the first two of these equations for $y_{\bar{u}}$ and $y_{\bar{v}}$, we obtain two very useful differentiation formulas

$$y_{\bar{u}} = -\frac{1}{2\lambda\varphi_v}(y_u - \lambda y_v), \qquad y_{\bar{v}} = \frac{1}{2\lambda\psi_v}(y_u + \lambda y_v). \tag{11}$$

Let us now substitute the expressions for y_{uu} and y_v from (10) into the first

of equations (2), and likewise substitute y_{vv} and y_u from (10) into the second of (2). System (2) goes over into another system of the form

$$
\begin{aligned}
y_{\bar u\bar u} &= \bar a y_{\bar v\bar v} + \bar b y_{\bar u} + \bar c y_{\bar v} + \bar d y,\\
y_{\bar u\bar v} &= \ast \qquad\; + \bar b' y_{\bar u} + \bar c' y_{\bar v} + \bar d' y,
\end{aligned} \tag{12}
$$

whose coefficients have the following values:

$$
\begin{aligned}
\bar a &= -\frac{\psi_v^2}{\varphi_v^2}, \qquad \bar d = -\frac{1}{2\lambda^2\varphi_v^2}(f + g\lambda^2), \qquad \bar d' = \frac{1}{4\lambda^2\varphi_v\psi_v}(f - g\lambda^2),\\
\bar b &= -\frac{1}{2\lambda^2\varphi_v^2}[(\lambda\lambda_v - \lambda_u + 2b - 2a'\lambda^3)\varphi_v + 2\lambda^2\varphi_{vv}],\\
\bar c &= -\frac{1}{2\lambda^2\varphi_v^2}[(\lambda\lambda_v + \lambda_u + 2b + 2a'\lambda^3)\psi_v + 2\lambda^2\psi_{vv}],\\
\bar b' &= \frac{1}{4\lambda^2\psi_v}(\lambda\lambda_v - \lambda_u + 2b + 2a'\lambda^3),\\
\bar c' &= \frac{1}{4\lambda^2\varphi_v}(\lambda\lambda_v + \lambda_u + 2b - 2a'\lambda^3).
\end{aligned} \tag{13}
$$

System (12) defines the same surface S_y as does system (2). But the parametric net for (12) is the arbitrary conjugate net (5), while the parametric net for (2) is the asymptotic net.

4. Calculation of the invariants and covariants

System (12) is fundamental in the theory of a conjugate net, as this theory was developed by G. M. Green. We shall write here for convenience of reference nine of Green's invariants and three of his covariants. The five fundamental invariants of (12) are*

$$
\begin{aligned}
\mathfrak{B}' &= \frac{1}{8\bar a}(4\bar a\,\bar b' + 2\bar c - \bar a_{\bar v}), \qquad \mathfrak{C}' = \frac{1}{8\bar a}(4\bar a\,\bar c' - 2\bar a\,\bar b + \bar a_{\bar u}),\\
\mathfrak{D} &= \bar d + \bar a\,\bar b'^2 - \bar c'^2 + \bar b'\bar c + \bar b\,\bar c' + \bar a\,\bar b'_{\bar v} - \bar c'_{\bar u},\\
\mathfrak{D}' &= \bar d' + \bar b'\bar c' - \frac{1}{4}\bar b_{\bar v} - \frac{1}{2}\bar c'_{\bar v}, \qquad \mathfrak{A} = \bar a.
\end{aligned} \tag{14}
$$

The two Laplace-Darboux invariants are given by†

$$
H = \bar d' + \bar b'\bar c' - \bar b'_{\bar u}, \; K = \bar d' + \bar b'\bar c' - \bar c'_{\bar v}, \tag{15}
$$

* Green, American Journal of Mathematics, vol. 37 (1915), p. 226. Cited as *First Memoir*.

† Green, *First Memoir*, p. 231–232.

and the two Weingarten invariants have the values*

$$W^{(\bar u)} = 2\bar b'_{\bar u} - \bar b_{\bar v} - \frac{\partial^2 \log \bar a}{\partial \bar u\, \partial \bar v}, \quad W^{(\bar v)} = 2\bar b'_{\bar u} - \bar b_{\bar v}\,. \tag{16}$$

The three covariants of (12) which we shall use, together with y, are†

$$\begin{aligned} \bar\rho &= y_{\bar u} - \bar c' y, \qquad \bar\sigma = y_{\bar v} - \bar b' y, \\ \bar\tau &= \bar a y_{\bar v \bar v} + \bar c y_{\bar v} - (\bar a\, \bar b'^2 + \bar b' \bar c + \bar a\, \bar b'_{\bar v} - \tfrac{1}{2}\mathfrak{D})y. \end{aligned} \tag{17}$$

We are now ready to calculate these invariants and covariants in terms of λ and the asymptotic parameters of S_y. Consider, for example, the invariant $\mathfrak{B}'$ as defined by the first of equations (14). We substitute therein the expressions for $\bar a$, $\bar b'$, $\bar c$ given by (13). In order to calculate $\bar a_{\bar v}$, we use the second of equations (11) as a differentiation formula. And so, in general, substituting the values of the coefficients (13) into Green's formulas for the invariants and covariants of system (12), and using (11) whenever it is necessary to calculate a derivative with respect to $\bar u$ or $\bar v$, *we obtain the following results, which express invariants and covariants of the arbitrary conjugate net* (5) *in terms of* λ *and the asymptotic parameters of* S_y *defined by* (2):

$$\mathfrak{A} = -\frac{\psi_v^2}{\varphi_v^2}, \qquad \mathfrak{B}' = \frac{1}{2\lambda^2\psi_v}(b + a'\lambda^3), \qquad \mathfrak{C} = \frac{1}{2\lambda^2\varphi_v}(b - a'\lambda^3),$$

$$\begin{aligned} \mathfrak{D} = \frac{1}{8\lambda^4\varphi_v^2}(2\lambda\lambda_{uu} &- 3\lambda_u^2 - 2\lambda^3\lambda_{vv} + \lambda^2\lambda_v^2 - 4a'_u\lambda^4 - 4\, b_v\lambda^2 \\ &- 12\, a'^2\lambda^6 - 12b^2 - 4\lambda^2 f - 4\lambda^4 g), \end{aligned}$$

$$\begin{aligned} \mathfrak{D}' = \frac{1}{16\lambda^4\varphi_v\psi_v}(-2\lambda\lambda_{uu} &+ 3\lambda_v^2 - 2\lambda^2\lambda_{uv} + 2\lambda\lambda_u\lambda_v - 2\lambda^3\lambda_{vv} + \lambda^2\lambda_v^2 \\ &- 4a'^2\lambda^6 + 4b^2 + 4a'\lambda^3\lambda_u + 4b\lambda\lambda_v + 4\lambda^2 f - 4\lambda^4 g), \end{aligned}$$

$$\begin{aligned} H = \frac{1}{8\lambda^4\varphi_v\psi_v}&(-\lambda\lambda_{uu} + \frac{3}{2}\lambda_u^2 - \lambda^3\lambda_{vv} + \frac{1}{2}\lambda^2\lambda_v^2 + 2a'_u\lambda^4 - 2b_v\lambda^2 \\ &+ 4a'\lambda^3\lambda_u - 2a'^2\lambda^6 + 4b\lambda\lambda_v + 2b^2 + 2\lambda^2 f - 2\lambda^4 g + 2\lambda^2\lambda_{uv} \\ &- 2\lambda\lambda_u\lambda_v - 2a'_v\lambda^5 + 2b_u\lambda - 4b\lambda_u - 4a'\lambda^4\lambda_v), \end{aligned} \tag{18}$$

$$W^{(\bar u)} = \frac{1}{2\lambda^3\varphi_v\psi_v}(\lambda\lambda_{uv} - \lambda_u\lambda_v - 2a'_u\lambda^3 - 2a'\lambda^2\lambda_u - 2b\lambda_v + 2b_v\lambda),$$

$$\bar\rho = -\frac{1}{4\lambda^2\varphi_v}[2\lambda y_u - 2\lambda^2 y_v + (\lambda\lambda_v + \lambda_u + 2b - 2a'\lambda^3)y],$$

$$\begin{aligned} \bar\tau = \frac{1}{4\lambda^4\varphi_v^2}\Big[&-2\lambda^3 y_{uv} + (\lambda^2\lambda_v - 2b\lambda)y_u - (\lambda^2\lambda_u + 2a'\lambda^5)y_v \\ &+ \left(\frac{1}{2}\lambda\lambda_u\lambda_v + 3a'\lambda^4\lambda_v + a'_v\lambda^5 - 3b\lambda_u + b_u\lambda + 6a'b\lambda^3\right)y\Big]. \end{aligned}$$

---

* Green, American Journal of Mathematics, vol. 38 (1916), p. 311–312. Cited as *Second Memoir*.

† Green, *First Memoir*, p. 230; *Second Memoir*, p. 292.

The omitted formulas for K, $W^{(\bar{v})}$, $\bar{\sigma}$ may be obtained from the formulas for H, $W^{(\bar{u})}$, $\bar{\rho}$, respectively, by changing the sign of λ and at the same time interchanging φ_v and ψ_v.

It will be observed that the functions φ and ψ enter the final form of the invariants and covariants only as extraneous factors whose value is immaterial.*

5. Isothermally conjugate nets

The parametric conjugate net of system (12) is isothermally conjugate if, and only if,†

$$\frac{\partial^2 \log \bar{a}}{\partial \bar{u}\, \partial \bar{v}} = 0.$$

Taking the value of $\bar{a}$ from (13) and applying the differentiation formulas (11), we readily find

$$\frac{\partial^2 \log \bar{a}}{\partial \bar{u}\, \partial \bar{v}} = -\frac{1}{\lambda \varphi_v \psi_v} \frac{\partial^2 \log \lambda}{\partial u\, \partial v}. \tag{19}$$

Therefore the conjugate net (5) is isothermally conjugate if, and only if, $\lambda = U_1 V_1$, where U_1 is a function of u alone and V_1 of v alone.

When the net (5) is isothermally conjugate, we may make a transformation of the group (4) which will reduce λ to unity. It is sufficient, in fact, to choose the arbitrary functions U and V of (4) so as to satisfy the conditions

$$U' = U_1, \qquad V'V_1 = 1.$$

Therefore *the differential equation defining an isothermally conjugate net on S_y may be reduced to the form*

$$dv^2 - du^2 = 0.$$

6. Pencils of conjugate nets

Let us consider a fundamental conjugate net

$$dv^2 - \lambda^2 du^2 = 0, \tag{5 bis}$$

and let us consider also the one-parameter family of conjugate nets defined by

$$dv^2 - \lambda^2 h^2 du^2 = 0, \tag{20}$$

---

* Cf. Green, *First Memoir*, p. 239.

† Green, *Second Memoir*, p. 320. Also Wilczynski, American Journal of Mathematics, vol. 42 (1920), p. 211. Cited as W. (1920).

where h^2 is an arbitrary constant, with respect to u and v, but is not zero. To any particular value of h^2 there corresponds a net of the family (20); this net has the property that at every surface point its tangents form with the tangents of the fundamental net (5) the same cross ratio, namely $(h-1)^2/(h+1)^2$. Such a one-parameter family of conjugate nets has been called by Wilczynski a pencil of conjugate nets.*

We shall state two useful properties of pencils of conjugate nets. First, the two nets of a pencil which correspond to the parameter-values h^2 and $-h^2$ are associate to each other in the sense of Green†, the tangents of each net separating the tangents of the other net harmonically at every surface point. Second, the nets of a pencil corresponding to the parameter-values h^2 and $1/h^2$ have the property that at every surface point the tangents of each net are the harmonic reflections of the tangents of the other net in the tangents of the fundamental net.

It is clear that the totality of curves in all the nets of a pencil form a two-parameter family of curves. If we solve the equation (20) of the pencil for the constant h, and then write the total derivative with respect to u of both members, we obtain the second order differential equation which defines the curves of a pencil of conjugate nets on S_y,

$$\frac{d^2v}{du^2} - \frac{\lambda_u}{\lambda}\frac{dv}{du} - \frac{\lambda_v}{\lambda}\left(\frac{dv}{du}\right)^2 = 0. \tag{21}$$

7. Ray point-cubic, ray conic, and their duals

We shall next consider some loci which are intimately connected with the notion of a pencil of conjugate nets.

The covariants $\bar{\rho}$ and $\bar{\sigma}$ define the Laplace transformations‡ of the fundamental conjugate net (5). The points $P_{\bar{\rho}}$ and $P_{\bar{\sigma}}$ lie in the tangent plane of S_y at P_y, and the locus of $P_{\bar{\rho}}$, as u and v vary, is the second sheet of the focal surface of the congruence of tangents to the curves $\bar{v}$ = const. on S_y. $P_{\bar{\sigma}}$ is similarly related to the curves $\bar{u}$ = const. on S_y.

Let us consider a pencil (20) of conjugate nets, and let us denote the covariants which determine the Laplace transformations of an arbitrary net of this pencil by $\bar{\rho}_h$ and $\bar{\sigma}_h$. Wilczynski has shown§ that, *as this arbitrary net varies over all the nets of the pencil, the locus of the points* $\bar{\rho}_h$ *and* $\bar{\sigma}_h$ *corresponding to a point* P_y *is a cubic curve in the tangent plane at* P_y. Since we shall wish to make use of this curve, we shall give here a new and very brief derivation of its equation.

* W. (1920), p. 216.
† Green, *Second Memoir*, p. 313. Also W. (1920), p. 218.
‡ Green, *Second Memoir*, p. 308.
§ Wilczynski, *Oral Communication*, Dec., 1920.

In the formulas for $\bar{\rho}$ and $\bar{\sigma}$ as given in (18), let us drop extraneous factors, and replace λ by λh. Thus we obtain

$$
\begin{aligned}
\bar{\rho}_h &= \left(y_u + \frac{1}{2}\frac{\lambda_u}{\lambda}y\right) - \lambda h\left(y_v - \frac{1}{2}\frac{\lambda_v}{\lambda}y\right) + \left(\frac{b}{\lambda h} - a'\lambda^2 h^2\right)y, \\
\bar{\sigma}_h &= \left(y_u + \frac{1}{2}\frac{\lambda_u}{\lambda}y\right) + \lambda h\left(y_v - \frac{1}{2}\frac{\lambda_v}{\lambda}y\right) - \left(\frac{b}{\lambda h} + a'\lambda^2 h^2\right)y.
\end{aligned} \tag{22}
$$

Let us define ρ and σ by the formulas

$$
\rho = y_u + \frac{1}{2}\frac{\lambda_u}{\lambda}y, \qquad \sigma = y_v - \frac{1}{2}\frac{\lambda_v}{\lambda}y, \tag{23}
$$

and let us choose the triangle $y\rho\sigma$ as a local triangle of reference for the tangent plane. Then the coördinates of the point $\bar{\rho}_h$, referred to the triangle $y\rho\sigma$, may be taken as

$$
x_1 = \frac{b}{\lambda h} - a'\lambda^2 h^2, \qquad x_2 = 1, \qquad x_3 = -\lambda h. \tag{24}
$$

When we eliminate h from (24) and make the result homogeneous in the usual way, *we obtain the desired equation of the locus of the point* $\bar{\rho}_h$, *in the form*

$$
a'x_3^3 + bx_2^3 + x_1x_2x_3 = 0. \tag{25}
$$

The locus of the point $\bar{\sigma}_h$ is the same curve. This curve has a node at the point $(1, 0, 0)$, which is the point P_y.

The nodal cubic has three collinear inflexion points, whose coördinates are easily shown to be

$$
\left(0, 1, -\sqrt[3]{\frac{b}{a'}}\right), \qquad \left(0, 1, -\omega\sqrt[3]{\frac{b}{a'}}\right), \qquad \left(0, 1, -\omega^2\sqrt[3]{\frac{b}{a'}}\right), \tag{26}
$$

where ω is a complex cube root of unity. The line on which these inflexion points of the nodal cubic lie has been called* the *flex-ray* of the point P_y. Its equation is $x_1 = 0$, so that our points P_ρ and P_σ are seen to be the points where the flex-ray of P_y crosses the asymptotic tangents of P_y. The directions from P_y to the three inflexion points are given by

$$
a'\,dv^3 + b\,du^3 = 0. \tag{27}
$$

The tangents with these directions have been called Darboux tangents, and the curves on S_y defined by (27) have been called the Darboux curves.†

* Wilczynski, *Oral Communication*, Dec., 1920.

† Green, *Congruences*, p. 142.

The flex-rays of all points P_y on S_y form a congruence. This congruence is, in the language of Green, a Γ-congruence, for which*

$$\alpha = \frac{1}{2}\frac{\lambda_v}{\lambda}, \qquad \beta = -\frac{1}{2}\frac{\lambda_u}{\lambda}. \tag{28}$$

Green's condition that the developables of a Γ-congruence correspond to a conjugate net on S_y is $\alpha_u - \beta_v = 0$. This condition, applied to the flex-ray congruence, gives us an elegant theorem of Wilczynski, that *the flex-ray curves form a conjugate net if, and only if, the fundamental conjugate net is isothermally conjugate.*† Moreover, we may obtain from (28) a characterization of all Γ-congruences for which $\alpha_u + \beta_v = 0$, as those Γ-congruences which are able to serve as flex-ray congruences for pencils of conjugate nets on the surface.

Wilczynski has shown‡ that *the space dual of the nodal cubic is a cone of the third class, called the axis cone,* which is enveloped by the osculating planes of the curves of an arbitrary net of the fundamental pencil, as this net varies over all the nets of the pencil. The osculating plane of a curve $\bar{v}$ = const. on S_y is determined by the points y, $y_{\bar{u}}$, $y_{\bar{u}\bar{u}}$. Calculating the value of $y_{\bar{u}\bar{u}}$ by means of the first of equations (11), dropping extraneous factors, and replacing λ by λh, we find that the osculating plane of an arbitrary curve $\bar{v}$ = const. of the fundamental pencil is determined by the points

$$y, \qquad y_u - \lambda h y_v, \quad z + a'\lambda h y_u + \frac{b}{\lambda h}\, y_v,$$

where we have placed

$$z = y_{uv} - \frac{1}{2}\frac{\lambda_v}{\lambda} + \frac{1}{2}\frac{\lambda_u}{\lambda}\, y_v. \tag{29}$$

Let us choose the tetrahedron $y\rho\sigma z$ as a local tetrahedron of reference. Referred to this tetrahedron, the coördinates of our three points are

$$(1, 0, 0, 0), \qquad (*, 1, -\lambda h, 0), \qquad \left(*, a'\lambda h, \frac{b}{\lambda h}, 1\right),$$

the omitted coördinates being immaterial for our purposes. And therefore the equation in point coördinates of the osculating plane of $\bar{v}$ = const. is

$$\lambda h x_2 + x_3 - \left(\frac{b}{\lambda h} + a'\lambda^2 h^2\right)x_4 = 0. \tag{30}$$

* Green, *Congruences*, p. 86.
† Wilczynski, *Oral Communication*, Dec., 1920.
‡ Wilczynski, *Oral Communication*, Dec., 1920.

The coördinates of this plane may be taken as

$$(31) \qquad u_1 = 0, \qquad u_2 = \lambda h, \qquad u_3 = 1, \qquad u_4 = -\left(\frac{b}{\lambda h} + a'\lambda^2 h^2\right).$$

When we eliminate h from (31) we find *the equation of the axis cone*

$$(32) \qquad a'u_2^3 + bu_3^3 + u_2u_3u_4 = 0.$$

This cone has three cusp-planes which intersect in a line called by Wilczynski the *cusp-axis* of P_y.* This is the line joining the points (1,0,0,0) and (0,0,0,1). Therefore the points P_y and P_z determine the cusp-axis. Reference to (28) and (29) will show that *the cusp-axis is the Green-reciprocal of the flex-ray.*

The cusp-axes of all the points P_y of the surface S_y form a congruence, which is, in the language of Green, a Γ'-congruence. The developables of the cusp-axis congruence cut the surface in a net of curves called the cusp-axis curves. In order to obtain the differential equation of these curves, we may use Green's formula for the Γ'-curves of an arbitrary Γ'-congruence,† namely

$$(f + \beta^2 + \beta_u - 2b\alpha + 2b_v)du^2 + (\beta_v - \alpha_u)du\,dv - (g + \alpha^2 + \alpha_v - 2a'\beta + 2a'_u)dv^2 = 0.$$

Substituting herein the values of α and β which are given by (28), we find *the differential equation of the cusp-axis curves*

$$(32a) \qquad \begin{aligned} &\left[\frac{1}{2}\frac{\lambda_{uu}}{\lambda} - \frac{3}{4}\left(\frac{\lambda_u}{\lambda}\right)^2 + b\frac{\lambda_v}{\lambda} - 2b_v - f\right]du^2 + \frac{\partial^2 \log \lambda}{\partial u\,\partial v}\,du\,dv \\ &+ \left[\frac{1}{2}\frac{\lambda_{vv}}{\lambda} - \frac{1}{4}\left(\frac{\lambda_v}{\lambda}\right)^2 + a'\frac{\lambda_u}{\lambda} + 2a'_u + g\right]dv^2 = 0. \end{aligned}$$

These curves form a conjugate net if, and only if,

$$\frac{\partial^2 \log \lambda}{\partial u\,\partial v} = 0.$$

Thus we obtain a theorem of Wilczynski: *The cusp-axis curves form a conjugate net if, and only if, the fundamental net is isothermally conjugate.*‡

The line joining $P_{\bar\rho}$ and $P_{\bar\sigma}$ has been called the ray of P_y.§ The ray of an arbitrary net of our fundamental pencil joins the points $\bar\rho_h$ and $\bar\sigma_h$. As the

* Wilczynski, *Oral Communication*, Dec., 1920.

† Green, *Congruences*, p. 90.

‡ Wilczynski, *Oral Communication*, Dec., 1920.

§ Wilczynski, *The general theory of congruences*, these Transactions, vol. 16 (1915), p. 317. Cited as W. (1915).

arbitrary net varies over all the nets of the pencil, its ray envelopes a curve in the tangent plane, which we shall show to be a conic and shall call the *ray-conic* of the point P_y. In order to find the equation of the ray-conic, we observe that any point η on the ray is given by an expression of the form

$$\eta = \omega_1 \bar{\rho}_h + \omega_2 \bar{\sigma}_h.$$

When h varies over its range, the point η describes a curve, and a point on the tangent to this curve is given by $d\eta/dh$. The point η is the point of contact of the ray with its envelope if, and only if, the point $d\eta/dh$ lies on the ray itself. Therefore we wish to determine the ratio $\omega_1 : \omega_2$ so that $d\eta/dh$ shall be expressible as a linear combination of $\bar{\rho}_h$ and $\bar{\sigma}_h$. The required value of this ratio is found to be

$$\omega_1 : \omega_2 = (b - a'\lambda^3 h^3) : (b + a'\lambda^3 h^3).$$

Therefore the point of contact of the ray with its envelope is

$$(b - a'\lambda^3 h^3)\bar{\rho}_h + (b + a'\lambda^3 h^3)\bar{\sigma}_h = 2b\rho + 2a'\lambda^4 h^4 \sigma - 4a'b\lambda^2 h^2 y.$$

Referred to the local triangle $y\rho\sigma$, the coördinates of this point may be taken as

$$x_1 = -2a'b\lambda^2 h^2, \qquad x_2 = b, \qquad x_3 = a'\lambda^4 h^4. \tag{33}$$

When we eliminate h *we obtain the equation of the ray-conic in the form*

$$4\,a'bx_2x_3 = x_1^2. \tag{34}$$

The asymptotic tangents of P_y are tangent to the ray-conic at P_ρ and P_σ. The flex-ray, joining P_ρ and P_σ, is the polar of P_y with respect to the ray-conic. The ray-conic (34) and the nodal cubic (25) are tangent to each other at three points, whose coördinates are obtained by taking the three values of the cube-roots indicated in the formulas

$$x_1 = -2(\sqrt[3]{a'b})^2, \qquad x_2 = \sqrt[3]{a'}, \qquad x_3 = \sqrt[3]{b}.$$

The directions from P_y to these points of tangency are given by

$$a'dv^3 - bdu^3 = 0. \tag{35}$$

The curves on S_y which are defined by (35) have been called the Segre curves.* The Segre curves and the Darboux curves (27) may be grouped in pairs to form three conjugate nets. These three nets belong to a pencil, which may be

* Green, *Congruences*, p. 142.

called *the Segre-Darboux pencil.* For this pencil $a'\lambda^3 + b = 0$ or $a'\lambda^3 - b = 0$ according as the curves $\bar{u} =$ const. or the curves $\bar{v} =$ const. are the Darboux curves. In Green's notations the corresponding conditions are $\mathfrak{B}' = 0$ or $\mathfrak{C}' = 0$.

The space dual of the ray of a point is the axis of the point.* The axis of the point P_y, relative to a conjugate net on S_y, has been defined to be the line of intersection of the osculating planes at P_y of the two curves of the net that pass through P_y. The axis of P_y, relative to our fundamental conjugate net (5), is determined by the points $P_{\bar{\tau}}$ and P_y, where $\bar{\tau}$ is the covariant given in the last one of formulas (18).† Replacing therein λ by λh, and making use of (29), we may write

$$\bar{\tau}_h = z + \frac{b}{\lambda^2 h^2} y_u + a'\lambda^2 h y_v + (?)y,$$

the coefficient of y being immaterial for our purposes.

The dual of the ray-conic of a pencil of conjugate nets is a quadric cone, which is generated by the axis of P_y relative to an arbitrary net of the pencil, as this net ranges over the pencil. The point $\bar{\tau}_h$ describes a curve on the cone, and the point $d\tau_h/dh$ is on the tangent to this curve.

Therefore the tangent plane of the axis quadric-cone is determined by the three points y, τ_h, $d\tau_h/dh$. The coördinates of these three points, referred to the local tetrahedron $y\rho\sigma z$ may be taken as

$$(1, 0, 0, 0), \left(*, \frac{b}{\lambda^2 h^2}, a'\lambda^2 h^2, 1\right), \left(*, -\frac{b}{h^2 h^2}, a'\lambda^2 h^2, 0\right),$$

the value of the omitted coördinates being immaterial for our purposes. Therefore the equation in point coördinates of the enveloping plane of the axis quadric-cone is

$$a'\lambda^4 h^4 x_2 + bx_3 - 2a'b\lambda^2 h^2 x_4 = 0. \tag{36}$$

The coördinates of this plane may be taken as

$$u_1 = 0, \qquad u_2 = a'\lambda^4 h^4, \qquad u_3 = b, \qquad u_4 = -2a'b\lambda^2 h^2. \tag{37}$$

Eliminating h, we find that *the equation in plane coördinates of the axis quadric-cone is*

$$4a'bu_2u_3 = u_4^2. \tag{38}$$

* W. (1915), p. 316.

† Green, *Second Memoir*, p. 292.

The loci which we have considered in this section lend themselves very readily to applications of the theory of union curves and adjoint union curves. Moreover, these loci themselves suggest certain generalizations and extensions of this theory, which lead to interesting results. But we shall not enter into a discussion of these questions in this paper.

8. Harmonic nets

Those conjugate nets for which the invariant $\mathfrak{D}$ vanishes have been called harmonic nets.* Such nets possess a number of harmonic properties, one of them being† that only for harmonic nets do the foci of the ray of each surface point separate the corresponding Laplace transform points harmonically.

We shall confine our discussion of harmonic nets to the subject of the number of harmonic nets in a pencil. Let us denote by $\mathfrak{D}_h$ the invariant $\mathfrak{D}$ for an arbitrary net of a fundamental pencil (20). Then, replacing λ by λh in the formula for $\mathfrak{D}$ in (18), we obtain, except for an extraneous factor,

$$\begin{aligned}(39)\qquad \mathfrak{D}_h = 12a'^2\lambda^6h^6 &+ (2\lambda^3\lambda_{vv} - \lambda^2\lambda_v^2 + 4a_u'\lambda^4 + 4\lambda^4 g)h^4 \\ &+ (-2\lambda\lambda_{uu} + 3\lambda_u^2 + 4b_v\lambda^2 + 4\lambda^2 f)h^2 + 12b^2.\end{aligned}$$

Since this invariant is an even function of λ and of h, let us place $\lambda^2 = \mu$, $h^2 = k$, and write (39) in the form

$$\begin{aligned}(40)\qquad \mathfrak{D}_k = 48\, a'^2\mu^4k^3 &+ \mu[16\mu^2(a_u' + g) + 4\mu\mu_{vv} - 3\mu_v^2]k^2 \\ &+ [16\,\mu^2(b_v + f) - 4\mu\mu_{uu} + 5\mu_u^2]k + 48\, b^2\mu.\end{aligned}$$

The harmonic nets in the fundamental pencil will be given by constant values of k which are roots of the equation $\mathfrak{D}_k = 0$. If there are more than three harmonic nets in the pencil, the equation $\mathfrak{D}_k = 0$, regarded as a cubic in k, has more than three roots and becomes an identity. Every net of the pencil is harmonic. Such pencils as consist entirely of harmonic nets will be called *harmonic pencils*.

For every net of a harmonic pencil, the four coefficients in the equation $\mathfrak{D}_k = 0$ vanish. From the vanishing of the first and last coefficients we obtain $a' = b = 0$. Therefore, *the sustaining surface of a harmonic pencil is a quadric*. The integrability conditions (3) show us that we have $g_u = f_v = 0$, and by a transformation of the type (4) we may reduce both f and g to zero. Then the vanishing of the other two coefficients shows that the fundamental net of the pencil is restricted by the conditions

$$(41)\qquad 4\mu\mu_{vv} - 3\,\mu_v^2 = 0, \qquad 4\mu\mu_{uu} - 5\,\mu_u^2 = 0.$$

---

* W. (1920), p. 215.

† Green, *First Memoir*, p. 243.

The general solution of equations (41) is of the form

$$\mu = (c_1 + c_2 v)^4/(c_3 + c_4 u)^4.$$

Reference to section (5) shows that *the fundamental net is isothermally conjugate*, and by a theorem of Wilczynski*, *every net of the pencil is isothermally conjugate*, Such pencils may be called *isothermally conjugate harmonic pencils*. We have proved the theorem:

If there are more than three harmonic nets in a pencil, the pencil is an isothermally conjugate harmonic pencil on a quadric surface.†

If there are exactly three harmonic nets in a pencil, the equation $\mathfrak{D}_k = 0$ has three constant roots, k_1, k_2, k_3. If we write

$$C_1 = k_1 + k_2 + k_3, \qquad C_2 = k_1 k_2 + k_2 k_3 + k_3 k_1, \qquad C_3 = k_1 k_2 k_3,$$

we obtain the following restrictions on the coefficients of the equation $\mathfrak{D}_k = 0$:

$$\begin{aligned} 16\mu^2(a'_u + g) + 4\mu\mu_{vv} - 3\,\mu_v^2 &= -\,48\,C_1 a'^2\mu^3, \\ 16\mu^2(b_v + f) - 4\mu\mu_{uu} + 5\mu_u^2 &= +\,48\,C_2 a'^2\mu^4, \\ C_3 a'^2\mu^3 + b^2 &= 0. \end{aligned} \tag{42}$$

The last of these equations shows that, *if a pencil contains exactly three harmonic nets, the pencil is the Segre-Darboux pencil*, which was defined in §7. The other two of equations (42) then *restrict the surface* sustaining the pencil.

9. Nets composed of plane curves

The conditions that a net may consist entirely of plane curves are‡

$$W^{(\bar{u})} + K = 0, \qquad W^{(\bar{v})} + H = 0. \tag{43}$$

We shall restrict our discussion to a consideration of the number of such nets which can be contained in a pencil.

Considering the pencil (20) as fundamental, we calculate the invariants H, K, $W^{(\bar{u})}$, $W^{(\bar{v})}$ for an arbitrary net of this pencil by replacing λ by λh in the formulas for these invariants in (18). Then if, as in §8, we let $\lambda^2 = \mu$ and $h^2 = k$,

* W. (1920), p. 218.

† This result was stated by W. (1920) without proof.

‡ Green, *Second Memoir*, p. 304.

we find that the nets of plane curves, if there be any, of the pencil are given by constant values of k which satisfy simultaneously the two equations

$$
\begin{aligned}
&2(a_v'\mu^3 + a'\mu^2\mu_v)k^2 + (\mu\mu_{uv} - \mu_u\mu_v)k + 2(b\mu_u - b_u\mu) = 0, \\
(44)\qquad &16a'^2\mu^4k^3 + \mu[48a_u'\mu^2 + 16a'\mu\mu_u + 16g\mu^2 + 4\mu\mu_{vv} - 3\mu_v^2]k^2 \\
&- [48b_v\mu^2 - 16b\mu\mu_v + 16\mu^2 f - 4\mu\mu_{uu} + 5\mu_u^2]k - 16b^2\,\mu = 0.
\end{aligned}
$$

Using the method of § 8, we may show without difficulty that, *if there are more than three nets of plane curves in one pencil, then the pencil is an isothermally conjugate pencil on a quadric surface. And if there are exactly three such nets in a pencil, the pencil is the Segre-Darboux pencil, and the surface is restricted by further conditions.* There is thus seen to be an analogy between the theory of harmonic nets and the theory of nets of plane curves, which promises to be a fruitful field for further study.

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PARALLEL MAPS OF SURFACES*

BY

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1. **Introduction.** Parallel maps, constituted by two surfaces in one to-one point correspondence with the normals at corresponding points parallel, have been frequently studied, especially in particular cases. The problem of the determination of maps of this type which are conformal, first proposed and solved by Christoffel,† has received, perhaps, the greatest attention. Certain aspects of equiareal maps have been considered by Guichard‡ and Razzaboni.§ Of importance in connection with infinitesimal deformations of a surface are the parallel maps such that the asymptotic lines on each surface correspond to a conjugate system on the other.|| The surfaces of a map of this kind, called by Bianchi *associate surfaces*, have been carefully investigated by Eisenhart.¶

The congruence of lines joining corresponding points of the surfaces of a parallel map have been found to have interesting properties. In particular, its two families of developables, according as they are distinct or coincident, intersect the surfaces in basic, i. e., corresponding, conjugate systems or in corresponding families of asymptotic lines.**

Despite these diverse investigations there is extant no general theory of parallel maps which is complete or thorough. To give such a theory is the primary purpose of this paper. Parallel maps are first classified as directly or inversely parallel and then as hyperbolic, elliptic, or parabolic, after the manner of classifying one-dimensional projective correspondences, and each map is characterized by an invariant I, analogous to the invariant of such a correspondence.

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† *Über einige allgemeine Eigenschaften der Minimumsflächen*, Journal für Mathematik, vol. 67 (1867), pp. 218–228; cf. Darboux, *Leçons*, 1st edition, vol. 2, p. 239; ibid., vol. 1, p. 326.

‡ *Sur les surfaces qui se correspondent avec parallèlisme des plans tangents et conservation des aires*, Comptes Rendus, vol. 136 (1903), pp. 151–153.

§ *Sulla rappresentazione equivalente di una superficie su di un' altra per parallelismo delle normali*, Rendiconti della Reale Accademia delle Scienze dell' Istituto di Bologna, ser. 6, vol. 9 (1912), pp. 45–59.

|| Bianchi, *Lezioni*, vol. 2, p. 10; German translation, 2d edition, p. 299.

¶ *Associate surfaces*, Mathematische Annalen, vol. 62 (1906), pp. 504–538.

** Cf. Darboux, *Leçons*, 1st edition, vol. 2, p. 234.

In the case of parabolic maps, for which $I = 1$, a second invariant, J, of purely metrical significance, is introduced.

The essential element in this classification is the invariant I. It has its most striking geometrical interpretation as the cross ratio in which the focal points of a line of the associated congruence divide the corresponding points of the surfaces. With its introduction the order in which the surfaces are taken becomes relevant, so that the map takes on the aspect of a transformation rather than that of a correspondence. If I is the invariant of the map of the one surface on the other, the reciprocal of I is the invariant of the map of the second surface on the first. The involutorial maps, for which $I = -1$, are those of associate surfaces of Bianchi.

As fundamental must be considered questions of existence of maps under given conditions. It is found that it is impossible to prescribe one surface of a non-parabolic map, the basic conjugate system on it, and the invariant I; nor is it possible to specify the spherical representation of the map, the curves on the sphere which are to represent the basic conjugate systems, and the invariant I. It is, however, possible to prescribe one of the surfaces and either the basic conjugate system on it or the invariant I, or to prescribe the spherical representation of the map, specifying the curves which are to represent the basic conjugate systems. In the case of parabolic maps, for which the corresponding families of asymptotic lines and the invariant J play the fundamental roles, similar results are found.

The classification, the discussion of the invariants, and the existence theorems are given in Parts I and IV. In Part II the theory is applied to maps of special type. Particular attention is accorded equiareal non-parabolic maps and parabolic maps of ruled surfaces. The consideration of the former gives rise to a generalization to arbitrary translation surfaces of the concept of associate minimal surfaces, with a well-rounded theory of parallel maps of translation surfaces as a result.

Part III is given to the investigation of the basic conjugate systems, the corresponding orthogonal systems (of a non-conformal map), and the isometrically mapped systems (of a non-isometric map). It is found that these systems, and certain angles associated with them, have interesting interrelationships, especially in the case of an equiareal map.

I. General theory

2. **Classification. The invariant** I. In studying parallel maps we shall restrict ourselves to the case in which the surfaces, S and S', and the correspondence between them are real and analytic. It is assumed further that S and S' are not developables and that the trivial case of two homothetic surfaces is excluded.

The normals at corresponding points shall be directed in the same sense. According, then, as corresponding directions of rotation about corresponding points are the same or opposite, the map shall be called *directly* parallel or *inversely* parallel.

Since in the neighborhoods of corresponding points the map is projective, there exists a unique system of curves on each surface such that the curves of the two systems correspond and have in corresponding points parallel tangents. According as the families of curves, C_1 and C_2, constituting the system on S, or the corresponding families, C_1' and C_2', of the system on S', are real and distinct, real and coincident, or conjugate-imaginary, the map shall be termed *hyperbolic*, *parabolic*, or *elliptic*. Evidently an inversely parallel map is always hyperbolic, whereas a directly parallel map may be of any one of the three types.

The invariant, I, of the projective correspondence established by a non-parabolic map between the directions of departure from corresponding points, P and P', shall be called the *invariant* of the map. More specifically, if the corresponding parallel directions at P (or P') are denoted, in a fixed order, by d_1 and d_2, and if d, at P, and d', at P', are arbitrary corresponding directions, I shall be defined as the cross ratio, constant for a given pair of points P, P', in which d and d' divide d_1 and d_2:

$$I = (d_1, d_2; d, d'). \tag{1}$$

The invariant I is a point function on each surface. For an inversely parallel map, it is negative; for a directly parallel map, it is positive, $\not\equiv 1$, or it is of the form $e^{\alpha i}$, $\alpha \not\equiv 2n\pi$, according as the map is hyperbolic or elliptic. Finally, it is agreed that, for a parabolic map, $I \equiv 1$.

The type of the function I determines the type of the map, unless $I \equiv -1$, when the map is either inversely parallel and hyperbolic or directly parallel and elliptic.

If I, as defined, is taken as the invariant of the map of S on S', the reciprocal of I, the cross ratio $(d_1, d_2; d', d)$ is the invariant of the (inverse) map of S' on S.

Inasmuch as the corresponding parallel families of curves, C_1, C_2 and C_1', C_2', constitute basic, i. e., corresponding, conjugate systems in the non-parabolic case and, in the parabolic case, coincide on each surface with one family of asymptotic lines (§1), whether these families are real or imaginary depends, in part, on whether the asymptotic lines on the two surfaces are real or imaginary. Accordingly, we consider the total curvatures of the surfaces.

For this purpose we define a directed element of area on a surface, $x = x(u, v)$:

$$x_1 = x_1(u, v), \qquad x_2 = x_2(u, v), \qquad x_3 = x_3(u, v).$$

According as the tangents to the parametric curves in the directions of increasing u and increasing v form with the directed normal a trihedral of the same disposition as the coördinate axes or of opposite disposition the element of area shall be positive or negative. In order that the direction cosines of the directed normal may always have their usual values, $1/H$ times the vector product of x_u and x_v, the determination H of the square root of $EG - F^2$ must be taken as positive in the first case and negative in the second. Then the formula,

$$dA = Hdudv,$$

always gives the directed element of area, dA. It follows that the corresponding directed element of area, $d\mathfrak{A}$, of the spherical representation is

$$d\mathfrak{A} = KHdudv = KdA.$$

Assume, now, that the map is established by assigning the same curvilinear coördinates to corresponding points, x and x', of the two surfaces:

$$x = x(u, v), \qquad x' = x'(u, v). \tag{2}$$

Since the surfaces have the same spherical representation, $d\mathfrak{A} = d\mathfrak{A}'$, or

$$KdA = K'dA'. \tag{3}$$

But the map is directly or inversely parallel according as $dAdA'$ is positive or negative. Hence KK' is positive in the first case, negative in the second.

The total curvatures of the two surfaces have the same or opposite signs according as the map is directly or inversely parallel.

If, in the case of a directly parallel map, the curvatures are both positive, the asymptotic lines on both surfaces are imaginary; therefore the families C_1, C_2, and C_1', C_2', are real and distinct and the map is hyperbolic. If, however, the curvatures are both negative, the map is hyperbolic or elliptic, according as the pairs of (real) asymptotic directions at corresponding points do not or do separate one another, and is parabolic if a direction of one pair is parallel to one of the other pair.

We can now give a complete classification of parallel maps:

	Curvatures of S, S'	*Type of map*	*Invariant*
Inversely parallel	Opposite in sign	Hyperbolic	$I < 0$,
Directly parallel	Both positive	Hyperbolic	$I > 0, \not\equiv 1$,
	Both negative	Hyperbolic	$I > 0, \not\equiv 1$,
		Parabolic	$I \equiv 1$,
		Elliptic	$I = e^{\alpha i}, \not\equiv 1$.

It is to be noted that, though the value of I in general determines the type of map, it does not always fix the signs of the curvatures of the surfaces.

3. **Geometric interpretation of the invariant** I. *Non-parabolic maps.* On the surfaces (2) let the corresponding parallel curves be parametric, C_1, C_1' the u-curves and C_2, C_2' the v-curves. Then scalar functions $\lambda(u, v)$, $\mu(u, v)$ exist such that

$$x_u' = \lambda x_u, \qquad x_v' = \mu x_v. \tag{4}$$

It is readily found that

$$I = \frac{\lambda}{\mu}. \tag{5}$$

In the hyperbolic case the parameters u and v and the functions λ and μ are real; the map is directly or inversely parallel according as $\lambda\mu > 0$ or $\lambda\mu < 0$. In the elliptic case u and v are conjugate-imaginary, and λ and μ are conjugate-imaginary functions of u and v.

Corresponding u-curves, C_1, C_1', are in Combescurian correspondence: the directions of their trihedrals at corresponding points are respectively parallel. Moreover, if the two curves are directed in the same sense,* corresponding directions of the two trihedrals can be similarly oriented. Then the ratio of the element of arc of C_1' to the element of arc of C_1 at corresponding points is λ. This is also the ratio of the radii of curvature of C_1' and C_1, and of their radii of torsion, of geodesic curvature and geodesic torsion.† We shall call λ the *Combescurian ratio* of the curves C_1' and C_1. Similarly μ is defined as the Combescurian ratio of two corresponding curves C_2', C_2.

THEOREM 1. *The invariant I of a non-parabolic map is the quotient of the Combescurian ratios of corresponding parallel curves C_1', C_1 and of corresponding parallel curves C_2', C_2.*

It follows that I is equal to the ratio R_{C_1}'/R_{C_2}', of the radii of normal curvature of S' in the basic conjugate directions divided by thecorresponding ratio, R_{C_1}/R_{C_2}, formed for S. In particular, if the basic conjugate systems are the lines of curvature, I is equal to the ratio r_1'/r_2' of the radii of principal curvature of S' divided by the corresponding ratio r_1/r_2, formed for S. Consequently, *the invariant of the parallel map of a surface on its spherical representation is the ratio, r_2/r_1, of the principal curvatures.*

A quite different interpretation of I comes to light in considering the congruence formed by the lines joining corresponding points, P and P', of S and S'. That the developables of this congruence intersect S and S' in the basic conjugate systems (§1) follows from the equations

* In the hyperbolic case, for example, let C_1 be oriented in the direction of increasing u on the surface S; then C_1 is oriented in the same sense, which is that of increasing or decreasing u on the surface S', according as $\lambda > 0$ or $\lambda < 0$.

† It is assumed that C_1 and C_1' are not minimal. For application of the facts here given, cf., e. g., Karl Peterson, *Über Curven und Flächen*, pp. 48–50.

$$(6)\quad \frac{\partial}{\partial u}\,\frac{x' - \lambda x}{1 - \lambda} = \frac{\lambda_u}{(1-\lambda)^2}(x' - x), \qquad \frac{\partial}{\partial v}\,\frac{x' - \mu x}{1 - \mu} = \frac{\mu_v}{(1-\mu)^2}(x' - x),$$

which are readily deduced from (4). From (6) it is clear, further, that the focal points, F_1 and F_2, associated respectively with the developables $D_1 : v =$ const. and the developables $D_2 : u =$ const., divide the line-segment $P'P$ in the ratios $-\lambda$ and $-\mu$. Consequently, the cross ratio of P', P and F_1, F_2 is equal to I.

Theorem 2. *The invariant of a non-parabolic map equals the cross ratio in which corresponding points P' and P of the surfaces S' and S are divided by those focal points, F_1 and F_2, of the associated congruence which lie on $P'P$.**

Evidently λ and μ, as well as I, are invariant under a change of curvilinear coördinates and under a homothetic transformation of both surfaces. It is important to note, however, that, unlike I, they are not invariant under *different* homothetic transformations applied, one to S, the other to S'.

Parabolic maps. Let the surfaces be represented by (2), with the corresponding parallel families, A and A', of asymptotic lines as the u-curves and any other corresponding families of real curves as the v-curves. Real scalar functions $\lambda(u, v)$, $\nu(u, v)$ exist such that

$$(7)\qquad x'_u = \lambda x_u, \quad x'_v = \nu x_u + \lambda x_v.$$

The Combescurian ratio, λ, of curves A' and A is an invariant similar in type to the λ and μ of the non-parabolic case. The function ν depends on the choice of parameters.

The two families of developables of the related congruence coincide and intersect the surfaces in the asymptotic lines A and A'. The single focal point

* In particular, if S and S' are associate surfaces ($I = -1$), F_1, F_2 divide P, P' harmonically. This fact has been noted by Bianchi, loc. cit., German translation, p. 301.

If $\lambda \equiv 1$ (or $\mu \equiv 1$), corresponding parallel curves C_1, C_1' (or C_2, C_2') are congruent by a translation; the developables D_1 (or D_2) are cylinders, the segments $P'P$ on the rulings of a cylinder are all equal, and the focal points F_1 (or F_2) are at infinity. If $\lambda_u \equiv 0$ but $\lambda \not\equiv 1$ (or $\mu_v \equiv 0$ but $\mu \not\equiv 1$), corresponding curves C_1, C_1' (or C_2, C_2') are homothetic but not congruent; the developables D_1 (or D_2) are cones, the segments $P'P$ on the rulings of a cone are all divided by the vertex F_1 (or F_2) in the same ratio, and the locus of the points F_1 (or F_2) is a curve. If the map is elliptic, the first of the two cases considered is impossible and the second occurs only when both $\lambda_u \equiv 0$ and $\mu_v \equiv 0$.

In general, if $\lambda_u \equiv \mu_v \equiv 0$, but neither λ nor μ is constant, equations (4) can be written as $x'_u = vx_u$, $x'_v = ux_v$, where the original parameters have been replaced by $u_1 = \mu(u)$, $v_1 = \lambda(v)$, and the subscripts subsequently dropped. Hence the surfaces S and S' are represented by

$$x = \frac{\varphi(v) - f(u)}{u - v}, \qquad x' = \frac{u\varphi(v) - vf(u)}{u - v},$$

where $f(u)$ and $\varphi(v)$ are arbitrary real triples in the hyperbolic case and arbitrary conjugate-imaginary triples in the elliptic case.

If λ and μ are both constant, S and S' are translation surfaces; cf. §9.

on a line of the congruence divides the line-segment $P'P$ in the ratio $-\lambda$. Theorem 2 can be considered to hold in a limiting form.*

4. **Fundamental theorem for a non-parabolic map.** From equations (4) follows

$$(\lambda - \mu)x_{uv} + \lambda_v x_u - \mu_u x_v = 0. \tag{8}$$

But

$$x_{uv} = \left\{\begin{matrix}12\\1\end{matrix}\right\}x_u + \left\{\begin{matrix}12\\2\end{matrix}\right\}x_v.$$

Consequently,

$$(\lambda - \mu)\left\{\begin{matrix}12\\1\end{matrix}\right\} + \lambda_v = 0, \qquad (\lambda - \mu)\left\{\begin{matrix}12\\2\end{matrix}\right\} - \mu_u = 0, \tag{9}$$

or

$$\left(1 - \frac{1}{I}\right)\left\{\begin{matrix}12\\1\end{matrix}\right\} + \frac{\partial \log \lambda}{\partial v} = 0, \qquad (I - 1)\left\{\begin{matrix}12\\2\end{matrix}\right\} - \frac{\partial \log \mu}{\partial u} = 0. \tag{10}$$

Differentiating the first of these equations with respect to u, the second with respect to v, and adding, we obtain the condition

$$\frac{\partial^2 \log I}{\partial u \partial v} + \frac{\partial}{\partial u}\left[\left(1 - \frac{1}{I}\right)\left\{\begin{matrix}12\\1\end{matrix}\right\}\right] + \frac{\partial}{\partial v}\left[(I - 1)\left\{\begin{matrix}12\\2\end{matrix}\right\}\right] = 0 \tag{11}$$

on the invariant I and the differential coefficients of the first order of the surface S.

Given, conversely, a surface S referred to a conjugate system, and a function $I(u, v)$, such that (11) is satisfied. Then the equation $\lambda = I\mu$ and equations (10), in λ and μ, are compatible; from them, by means of a quadrature, λ and μ are determined uniquely except for the same multiplicative constant k. Since these values of λ and μ satisfy (9) and hence (8), equations (4) are integrable and determine x' uniquely except for the multiplicative constant k and an additive triple.†

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* The case, $\lambda_u \equiv 0$, in which it turns out that S and S' are ruled surfaces, is given special attention in §13.

† The values found for λ and μ are

$$\lambda = ke^M, \qquad M = \int\left[(I - 1)\left\{\begin{matrix}12\\2\end{matrix}\right\} + \frac{\partial \log I}{\partial u}\right] du + \left(\frac{1}{I} - 1\right)\left\{\begin{matrix}12\\1\end{matrix}\right\} dv,$$

$$\mu = ke^N, \qquad N = \int (I - 1)\left\{\begin{matrix}12\\2\end{matrix}\right\} du + \left[\left(\frac{1}{I} - 1\right)\left\{\begin{matrix}12\\1\end{matrix}\right\} - \frac{\partial \log I}{\partial v}\right] dv.$$

Hence

$$y = k\int e^M x_u du + e^N x_v dv.$$

Theorem 3. *If a conjugate system on a given surface S and a point function $I(u, v)$ on S are chosen, a necessary and sufficient condition that there exist a surface S' on which S is mapped by parallel normals with the given conjugate system basic and the given function I as the invariant, is that, when S is referred to the conjugate system, E, F, G, and I satisfy* (11). *The surface S' is then determined to within its homothetics and its point coördinates can be found by quadratures.*

It follows that, if a surface S and a conjugate system on it are given, there exist infinitely many surfaces S' on which S is mapped by parallel normals with the given conjugate system basic. The determination of the surfaces S' depends, when S is referred to the conjugate system, on the solution of (11) for I and on subsequent quadratures.

Equation (11) is invariant under a change of parameters of the form $u = u(u')$, $v = v(v')$ or of the form $u = u(v')$, $v = v(u')$, provided that in the latter case I is replaced by $1/I$. Accordingly, $\left\{ {12 \atop 1} \right\}$ and $\left\{ {12 \atop 2} \right\}$ can be thought of as the invariants of (11). If either is zero, the equation can be integrated by quadratures.

By means of the formulas*

$$\left\{ {12 \atop 1} \right\} = I \left\{ {12 \atop 1} \right\}', \qquad \left\{ {12 \atop 2} \right\} = \frac{1}{I} \left\{ {12 \atop 2} \right\}', \tag{12}$$

where the symbols with primes refer to the surface S', we deduce from (11) the equation

$$\frac{\partial^2 \log I}{\partial u \partial v} + \frac{\partial}{\partial u}\left[(I-1)\left\{ {12 \atop 1} \right\}'\right] + \frac{\partial}{\partial v}\left[\left(1 - \frac{1}{I}\right)\left\{ {12 \atop 2} \right\}'\right] = 0, \tag{13}$$

which is the basis for a theorem analogous to Theorem 3 when the *transformed* surface S', and a conjugate system and a point function $I(u, v)$ on it, are given.

By the use of (12) either (11) or (13) can be put into the symmetric form

$$\frac{\partial^2 \log I}{\partial u \partial v} + \left[\frac{\partial}{\partial u}\left\{ {12 \atop 1} \right\} - \frac{\partial}{\partial v}\left\{ {12 \atop 2} \right\}\right] - \left[\frac{\partial}{\partial u}\left\{ {12 \atop 1} \right\}' - \frac{\partial}{\partial v}\left\{ {12 \atop 2} \right\}'\right] = 0, \tag{14}$$

bearing on both S and S'. The quantities in the square brackets are the differences of the point invariants of the basic conjugate systems on S and S'. Since

* These formulas are most simply deduced by noting that

$$\left\{ {12 \atop 1} \right\} = \frac{(x_{uv} x_v | x_u x_v)}{H^2}, \qquad \left\{ {12 \atop 2} \right\} = \frac{(x_u x_{uv} | x_u x_v)}{H^2},$$

where $(x_{uv} x_v | x_u x_v)$, for example, is the scalar product of the vector products of x_{uv}, x_v and x_u, x_v.

these invariants are unchanged by a projective transformation of S and S',* (14) is invariant under such a transformation.

5. **The invariant J and the fundamental theorem for a parabolic map.** A parabolic projective transformation of a pencil of lines into itself, when the vertex of the pencil is chosen as the origin of a system of rectangular coördinates in the plane of the pencil and the double line is taken as the axis of x, is represented by an equation of the form

$$\frac{1}{\lambda'} - \frac{1}{\lambda} = J,$$

where λ and λ' are, respectively, the slopes of the given and transformed lines. The constant J, equal to the difference of the cotangents of the angles which the transformed and given lines make with the directed double line, is a metrical invariant of the transformation, and, taken with the double line, determines the transformation uniquely. The invariant of the inverse transformation is $-J$ and that of the identity, zero.

As the invariant $J(u, v)$ of a parabolic parallel map we define the difference of the cotangents of the angles which two corresponding curves, C and C', make at corresponding points with the parallel, similarly directed,† asymptotic lines, A and A', through these points:

$$J(u, v) = \cot(C', A') - \cot(C, A).$$

It is clear that J is invariant under different homothetic transformations applied, one to S, the other to S'.

THEOREM 4. *The invariant J of a parabolic map equals twice the difference of the cotangents of the angles which the non-corresponding asymptotic lines, $\overline{A}'$ and $\overline{A}$, make with the basic, similarly directed, asymptotic lines, A' and A*:

$$J(u, v) = 2[\cot(\overline{A}', A') - \cot(\overline{A}, A)].$$

It follows that two surfaces cannot correspond by a parabolic map so that the angles between the asymptotic lines at corresponding points are equal. In particular, *two minimal surfaces can never correspond by a parabolic map.*

From equations (7), taking as C and C' corresponding v-curves, we find that

$$J = \frac{\nu}{\lambda}\frac{E}{H}. \tag{15}$$

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* Voss, *Zur Theorie der Krümmung der Flächen*, Mathematische Annalen, vol. 39 (1891) (pp. 179–256), p. 196.

† Orient A in the direction of increasing u on the surface S and give to A' the same orientation.

In developing the conditions for the integrability of (7) we have, first, that

$$\nu x_{uu} + (\nu_u - \lambda_v)x_u + \lambda_u x_v = 0. \tag{16}$$

Hence

$$\nu\begin{Bmatrix}11\\1\end{Bmatrix} + \nu_u - \lambda_v = 0, \qquad \nu\begin{Bmatrix}11\\2\end{Bmatrix} + \lambda_u = 0. \tag{17}$$

These equations are equivalent to the equations

$$\frac{\nu}{\lambda}\left(\begin{Bmatrix}11\\1\end{Bmatrix} - \frac{\nu}{\lambda}\begin{Bmatrix}11\\2\end{Bmatrix}\right) + \frac{\partial}{\partial u}\left(\frac{\nu}{\lambda}\right) - \frac{\partial \log \lambda}{\partial v} = 0, \quad \frac{\nu}{\lambda}\begin{Bmatrix}11\\2\end{Bmatrix} + \frac{\partial \log \lambda}{\partial u} = 0, \tag{18}$$

from which follows, by differentiation and the introduction of J, the condition

$$\frac{\partial^2}{\partial u^2}\left(J\frac{H}{E}\right) + \frac{\partial}{\partial u}\left[J\frac{H}{E}\left(\begin{Bmatrix}11\\1\end{Bmatrix} - J\frac{H}{E}\begin{Bmatrix}11\\2\end{Bmatrix}\right)\right] + \frac{\partial}{\partial v}\left(J\frac{H}{E}\begin{Bmatrix}11\\2\end{Bmatrix}\right) = 0. \tag{19}$$

Given, conversely, a surface S, referred to one family of asymptotic lines as the u-curves, and a function $J(u, v)$, such that (19) is satisfied. Then equations (18) in λ, where ν/λ has been replaced by JH/E, are integrable and determine λ except for a multiplicative constant k. Since this value of λ and the resulting value $\nu = \lambda JH/E$ for ν satisfy (16), x' can be determined from (7) except for the constant multiplier k and an additive triple.

THEOREM 5. *If one family of asymptotic lines on a given surface S, of negative curvature, and a point function $J(u, v)$ on S are chosen, a necessary and sufficient condition that there exist a surface S' on which S is mapped by parallel normals with the given asymptotic lines basic and the given function $J(u, v)$ as the invariant is that, when S is referred to the asymptotic lines as the u-curves, E, F, G, and J satisfy* (19). *The surface S' is then determined to within its homothetics and its point coördinates can be found by quadratures.*

It is evident that there exist infinitely many surfaces S' on which a given surface S, of negative curvature, is mapped by parallel normals so that a chosen family of asymptotic lines on S is basic. When S is ruled and the given asymptotic lines are the rulings, i. e., when $\begin{Bmatrix}11\\2\end{Bmatrix} = 0$, the surfaces S' can be determined by quadratures; cf. §13.

By means of the formulas,

$$\frac{H}{E} = \frac{H'}{E'}, \quad \begin{Bmatrix}11\\1\end{Bmatrix} - J\frac{H}{E}\begin{Bmatrix}11\\2\end{Bmatrix} = \begin{Bmatrix}11\\1\end{Bmatrix}' + J\frac{H'}{E'}\begin{Bmatrix}11\\2\end{Bmatrix}', \quad \begin{Bmatrix}11\\2\end{Bmatrix} = \begin{Bmatrix}11\\2\end{Bmatrix}', \tag{20}$$

we could deduce from (19) an equation similar in form to (19), but involving only

E', F', G', and J, and thus obtain a theorem, analogous to Theorem 5, when the *transformed* surface S' is given.

II. Maps of special type

6. **Differential coefficients of the surfaces.** Throughout Parts II and III we assume that the map is established by equations (4) or (7) of §2, according as it is non-parabolic or parabolic.

From equations (4) for a *non-parabolic* map we find that*

$$(21)\qquad \begin{aligned} &E' = \lambda^2 E, && F' = \lambda\mu F, && G' = \mu^2 G, && H' = \lambda\mu H;\\ &e' = \lambda e, && f' = f = 0, && g' = \mu g, && h'^2 = \lambda\mu h^2;\\ &&& K' = \frac{1}{\lambda\mu}K. \end{aligned}$$

For a *parabolic* map we have, from (7):

$$(22)\qquad \begin{aligned} &E' = \lambda^2 E, && F' = \lambda\nu E + \lambda^2 F, && G' = \nu^2 E + 2\lambda\nu F + \lambda^2 G;\\ &e' = e = 0, && f' = \lambda f, && g' = \mu f + \lambda g;\\ &H' = \lambda^2 H, && h'^2 = \lambda^2 h^2, && K' = \frac{1}{\lambda^2}K. \end{aligned}$$

7. $I = -1$. **Associate surfaces of Bianchi.** In taking up special cases we consider, first, the involutory maps, for which $I = -1$.

Theorem 6. *The pairs of asymptotic directions at two corresponding points of the two surfaces of a parallel map separate one another harmonically if and only if $I = -1$.*

The relationship is clearly impossible in the parabolic case. In the non-parabolic case, since the asymptotic directions on S' are defined, according to (21), by $Iedu^2 + gdv^2 = 0$, the directions on S parallel to them are given by $edu^2 + Igdv^2 = 0$;† but these directions separate harmonically the asymptotic directions on S if and only if $I = -1$.

A necessary and sufficient condition that the surfaces of a non-parabolic map be associate surfaces of Bianchi (cf. §1) is that $eg' + e'g = 0$, i. e., by (21), that $(I + 1)eg = 0$, or, by virtue of initial assumptions (§2), that $I = -1$. But a parabolic map can never have the property in question. Consequently:

Theorem 7. *Two surfaces mapped by parallel normals are associate surfaces of Bianchi if and only if the invariant I of the map is -1.*

For the purpose of stating a more general characteristic property of involu-

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* The quantities e, f, g are the differential coefficients of the second order, and $h^2 = eg - f^2$.

† The direction at a point of S parallel to the direction du/dv at the corresponding point of S' is Idu/dv.

tory maps we agree to call two directions on S *cross-parallel* to the corresponding directions on S' if each is parallel to the one to which it does not correspond.

THEOREM 8. *If $I = -1$, each two directions on S which are in involution with the basic conjugate directions are cross-parallel to the corresponding directions on S'. If $I \neq -1$, a pair of directions is never cross-parallel to the corresponding pair.**

When $I = -1$, equations (11) and (13) become

$$\frac{\partial}{\partial u}\begin{Bmatrix}12\\1\end{Bmatrix} - \frac{\partial}{\partial v}\begin{Bmatrix}12\\2\end{Bmatrix} = 0, \qquad \frac{\partial}{\partial u}\begin{Bmatrix}12\\1\end{Bmatrix}' - \frac{\partial}{\partial v}\begin{Bmatrix}12\\2\end{Bmatrix}' = 0. \tag{23}$$

THEOREM 9. *The basic conjugate systems on two associate surfaces of Bianchi have equal point invariants.*

From Theorem 3 we obtain, when $I = -1$:

THEOREM 10. *If a conjugate system with equal point invariants on a given surface S is chosen, there exists a surface S' unique to within its homothetics, which is associate to S in the sense of Bianchi, so that the given conjugate system on S is basic.*

When S is referred to the conjugate system, the coördinates of S' are found to be

$$x' = k\int e^{-2\varphi}(x_u\,du - x_v\,dv),$$

where

$$\varphi = \int \begin{Bmatrix}12\\2\end{Bmatrix}du + \begin{Bmatrix}12\\1\end{Bmatrix}dv. \tag{24}$$

The converse of Theorem 9 is not true, as (14) shows. We can, however, prove the theorem:

THEOREM 11. *If the basic conjugate systems of a non-parabolic map have equal point invariants and I is constant, the surfaces are, in general, associate surfaces of Bianchi.*

For, in this case (11) reduces to

$$\frac{\partial}{\partial u}\begin{Bmatrix}12\\1\end{Bmatrix} + I\frac{\partial}{\partial v}\begin{Bmatrix}12\\2\end{Bmatrix} = 0,$$

which in conjunction with the first of equations (23) gives, in general, $I = -1$.

* In particular, the asymptotic directions on each surface of an involutory map are cross-parallel to the conjugate directions corresponding to them on the other surface. This fact and Theorems 9 and 10 are well known; cf., e. g., Eisenhart, *Differential Geometry*, pp. 380, 381.

An exception occurs when

$$\frac{\partial}{\partial u}\left\{\begin{matrix}12\\1\end{matrix}\right\} = 0, \qquad \frac{\partial}{\partial v}\left\{\begin{matrix}12\\2\end{matrix}\right\} = 0.$$

Then, according to (12), the corresponding equations hold for S' and it is found, after a suitable change of parameters, that the point coördinates of S and S' are, respectively, solutions of the equations:

$$\Theta_{uv} = \Theta_u + \Theta_v, \qquad \Theta_{uv} = \frac{1}{I}\Theta_u + I\,\Theta_v.$$

8. **Conjugate systems with equal point invariants.** From (14) we conclude the following:

THEOREM 12. *If the basic conjugate system on one surface of a non-parabolic map has equal point invariants, that on the other surface has also, if and only if I is of the form $U(u)/V(v)$.*

If, in Theorem 3, the conjugate system chosen on the given surface S has equal point invariants, condition (11) becomes

$$\frac{\partial^2 \log I}{\partial u \partial v} - \frac{\partial}{\partial u}\left[\frac{1}{I}\left\{\begin{matrix}12\\1\end{matrix}\right\}\right] + \frac{\partial}{\partial v}\left[I\left\{\begin{matrix}12\\2\end{matrix}\right\}\right] = 0. \tag{25}$$

If we demand, further, that the basic conjugate system on the surface S' to be determined also have equal point invariants, (25) is replaced by the requirement that I be of the form $U(u)/V(v)$ and satisfy the equation

$$\frac{\partial}{\partial u}\left(\frac{1}{I}\varphi_v\right) = \frac{\partial}{\partial v}\left(I\varphi_u\right),$$

where φ is defined by (24).

To determine all the surfaces S, to each of which there corresponds a surface S' by a parallel map with a prescribed invariant $I = U(u)/V(v)$ so that the basic conjugate systems on both surfaces have equal point invariants, it is necessary, first, to integrate the equation in φ:

$$\frac{\partial}{\partial u}\left(\frac{V}{U}\varphi_v\right) = \frac{\partial}{\partial v}\left(\frac{U}{V}\varphi_u\right),$$

and then to integrate the equation in x : $x_{uv} = \varphi_v x_u + \varphi_u x_v$.

9. **Translation surfaces.** If one of the two surfaces of a non-parabolic map is a translation surface whose generators form the basic conjugate system, then, by (12), the other is also a translation surface with its generators basic. According to (9) this case occurs if and only if $\lambda_v = 0$ and $\mu_u = 0$, i. e., if and only if

the Combescurian ratio of the basic curves C_1' and C_1 is constant along the basic curves C_2' and C_2, and vice versa. Evidently $I = \lambda(u)/\mu(v)$, or

$$\frac{\partial^2 \log I}{\partial u \partial v} = 0. \tag{26}$$

THEOREM 13. *Two translation surfaces can be mapped by parallel normals so that the generators correspond if and only if two non-congruent generators of one can be put simultaneously into Combescurian correspondence with two non-congruent generators of the other. If both correspondences are homothetic, the invariant of the map is constant, and conversely.**

To prove the converse last stated, we note that, if $I = \lambda(u)/\mu(v)$ is constant, λ and μ are constant.

THEOREM 14. *The only surfaces on which a given translation surface*

$$x = U(u) + V(v) \tag{27}$$

is mapped by parallel normals so that its generators are basic are the translation surfaces

$$x' = \int \lambda(u)U'(u)du + \int \mu(v)V'(v)dv, \tag{28}$$

where $\lambda(u)$ and $\mu(v)$ are arbitrary scalar functions, not zero.

For, in this case (11) reduces to (26) and (9) to $\lambda_v = 0$, $\mu_u = 0$.

10. **Conformal and isometric maps.** The following three theorems, whose content except in so far as it affects the value of I is well known,† are stated without proof for later reference.

THEOREM 15. *A directly parallel map is conformal if and only if the two surfaces are minimal surfaces. Conversely, any two minimal surfaces can be brought into a correspondence by parallel normals; the map is always directly conformal and elliptic and, when the minimal lines are parametric, I is of the form $U(u)/V(v)$.*

THEOREM 16. *A directly parallel map is isometric if and only if the surfaces are associate minimal surfaces. The map is always elliptic; its invariant is constant and has the value -1 when and only when the surfaces are adjoint.*

THEOREM 17. *An inversely parallel map is conformal if and only if the surfaces are associate surfaces of Bianchi whose lines of curvature correspond. The surfaces are isometric surfaces; that is, the lines of curvature on each form an isometric system.*

* Suitable changes must be made in the conditions imposed if the surfaces are doubly covered by single families of generators.

† Cf. the reference to Christoffel, §1. The proofs can be readily constructed by use of equations (21), (22) and §§ 7, 9.

According to Theorem 10, to a given isometric surface S there corresponds, by an inversely parallel conformal map, an isometric surface S' determined to within its homothetics:*

$$x' = k\int e^{-\varphi}(x_u du - x_v dv), \qquad \text{where} \qquad \varphi = \int \frac{\partial \log G}{\partial u} du + \frac{\partial \log E}{\partial v} dv.$$

If two isometric surfaces are mapped by parallel normals so that the lines of curvature correspond, I is of the form $U(u)/V(v)$; the map is not necessarily conformal (or inversely parallel). However, by Theorem 11, if we stipulate that I be constant, the map is, in general, conformal. The exceptional case occurs when E, G, E', G' are all functions of the form $U(u)/V(v)$; the linear element of the surface S, referred to its lines of curvature, can then be written

$$ds^2 = U(u)V(v)(du^2 + dv^2),$$

whereupon that of S' becomes

$$ds'^2 = k^2\, U^I\, V^{\frac{1}{I}}\, (I^2 du^2 + dv^2).$$

It is to be noted that surfaces of revolution are of the type in question.†

A necessary and sufficient condition that an inversely parallel map be isometric is, by (21), that $\lambda = \pm 1$, $\mu = \mp 1$, and $F = 0$. The surfaces are, then, translation surfaces with orthogonal generators and can be represented by equations of the form:

$$x = U(u) + V(v), \quad x' = \pm U(u) \mp V(v). \tag{29}$$

It can be shown that they are congruent cylinders and that the correspondence is equivalent in the one case to a reflection in a plane and a translation, and in the other to a reflection in a line.

11. **Equiareal non-parabolic maps**. A non-parabolic map is equiareal, by (21), if and only if the product, $\lambda\mu$, of the Combescurian ratios of corresponding parallel curves is $+1$ or -1, according as the map is directly or inversely parallel. From (21), or (3), we have also:

THEOREM 18. *A non-parabolic map is equiareal when and only when the total curvatures of the surfaces at corresponding points are equal or opposite, according as the map is directly or inversely parallel.*‡

* When the parameters on S are isometric, E and G are equal and the representation of S' reduces to that usually given; cf. Eisenhart, *Differential Geometry*, p. 388.

† Since a surface whose lines of curvature have equal point invariants is isometric, the results of §8 also are applicable here.

‡ This theorem is, of course, well known; cf. the references to Guichard and Razzaboni, §1. The remaining results of the section are believed to be new.

Let the surface S:

$$x = U(u) + V(v)$$

be a translation surface referred to its generators. If these are real, there are two distinct one-parameter families of translation surfaces which we shall call *associate* to S, namely the families

$$(A) \qquad x' = \lambda U(u) - \frac{1}{\lambda} V(v),$$

$$(B) \qquad x' = \lambda U(u) + \frac{1}{\lambda} V(v),$$

where in each case λ is an arbitrary real constant, $\neq 0$. The family (A) does not contain the given surface, whereas the family (B) does.

If the generators of S are conjugate-imaginary, the surfaces associate to it shall be defined by the representation

$$(C) \qquad x' = e^{\alpha i} U(u) + e^{-\alpha i} V(v),$$

where α is an arbitrary real constant.

A special case of the family (C) is that of surfaces associate to a minimal surface. As in that case, so here also, the path curve of a chosen point (u, v) of a variable surface of the family is an ellipse whose center is at the origin of coördinates, O. A path curve for the family (B) is a hyperbola whose center is at O and whose asymptotes are the lines through O and the points u and v of the curves $y = U(u)$, $z = V(v)$; the conjugate hyperbola is the corresponding path curve for the family (A).

If S is minimal, the surfaces (C) for which $\alpha = \pm \pi/2$ are known as its adjoints. Accordingly, we define, as the adjoints of an arbitrary translation surface S, the surfaces

$$x' = \pm U(u) \mp V(v), \quad \text{or} \quad x' = \pm iU(u) \mp iV(v),$$

according as the generators of S are real or imaginary.

The translation surfaces (27) and (28) are associate if and only if λ and μ of (28) are constants such that $\lambda\mu = \pm 1$; they are adjoint if and only if $\lambda = \pm 1$, $\mu = \mp 1$, or $\lambda = \pm i$, $\mu = \mp i$. Thus:

THEOREM 19. *A non-parabolic map is equiareal with constant invariant when and only when the surfaces are associate translation surfaces. It is equiareal with invariant* -1 *if and only if the surfaces are adjoint translation surfaces.*

If the map is hyperbolic, the surfaces have real generators and belong both to a family of type (B), or one to a family of type (B) and the other to the corresponding family of type (A), according as the map is directly or inversely

parallel. In the elliptic case the surfaces have imaginary generators and both belong to a family of type (C).

From this discussion and Theorem 18 we conclude the following:

THEOREM 20. *The total curvatures of two associate translation surfaces at corresponding points are equal, if the generators are imaginary; if the generators are real, they are equal or opposite, according as the surfaces belong to the same or different families.*

Associate translation surfaces constitute the only equiareal maps of translation surfaces such that the generators correspond. For, if $\lambda(u)$ and $\mu(v)$ of (28) satisfy $\lambda\mu = \pm 1$, λ and μ are constant.

12. **Summary of special maps for which $I = -1$.** It is instructive to collect from §§10, 11 the results which concern associate surfaces of Bianchi.

These surfaces, in the case of an inversely parallel map, are: (a) if the map is equiareal, adjoint translation surfaces with real generators; (b) if the map is isometric, adjoint translation surfaces with orthogonal generators, i. e., cylinders; (c) if the map is conformal, isometric surfaces.

If the map is directly parallel, the surfaces, in the three cases considered, are: (a) adjoint translation surfaces with conjugate-imaginary generators; (b) adjoint translation surfaces with minimal generators, i. e., adjoint minimal surfaces; (c) the same as in (b),—a directly conformal parallel map for which $I = -1$ is necessarily isometric.

13. **Parabolic maps of ruled surfaces.** The surfaces of a parabolic map are, by (16), ruled surfaces whose rulings correspond if and only if $\lambda_u = 0$, i. e., if and only if along each pair of basic asymptotic lines the Combescurian ratio is constant. From §3 we find also:

THEOREM 21. *The focal surface of the congruence associated with a parabolic map degenerates into a curve if and only if the surfaces of the map are ruled surfaces whose generators correspond.*

Then every developable of the congruence is a plane determined by two corresponding generators, and the line-segments joining corresponding points of these generators are all divided by the focal point in the plane in the same ratio, $-\lambda(v)$.

In deducing the equations of the general pair of ruled surfaces mapped by parallel normals so that the generators correspond, we distinguish two cases.

Case 1: $\lambda(v) \neq$ *const.* Introducing instead of u in the equations of a parabolic map a new parameter $\bar{u}$ such that $\partial\bar{u}/\partial u = \lambda'(v)u/\nu$, and subsequently dropping the bars, we find that (7) becomes*

$$x'_u = \lambda(v)x_u, \qquad x'_v = \lambda'(v)ux_u + \lambda(v)x_v,$$

* Throughout the paragraph we denote the derivatives of functions of a single variable, e. g., $\lambda(v)$, by using primes.

and that (16) reduces to $x_{uu} = 0$. It follows, then, that

$$x = \psi(v)u + \varphi(v), \qquad x' = \lambda(v)\psi(v)u + f(v), \tag{30}$$

where
$$f'(v) = \lambda(v)\varphi'(v).$$

Equations (30), *where* $\lambda(v)$ *is an arbitrary scalar function, not constant, and* $f(v)$, $\varphi(v)$, $\psi(v)$ *are arbitrary triples of functions such that* $f' \equiv \lambda\varphi'$, *define the general parabolic map of two ruled surfaces whose generators correspond, in the case that the Combescurian ratio of corresponding rulings is not constant.*

The map of S on S' is equivalent to what might be called a pseudo-homothetic transformation,* consisting of a Combescurian transformation of ratio $\lambda(v)$ of the directrix curve, $X = \varphi(v)$, of S into the directrix curve, $X' = f(v)$, of S', and a homothetic transformation of ratio $\lambda(v)$ carrying the ruling through X into the ruling through X' so that the point X goes into the point X'.

Along the directrix curves the map has a singular behavior, in that in the neighborhoods of corresponding points of these curves it is equivalent, to a first approximation, to a homothetic transformation of ratio $\lambda(v)$; in particular, corresponding directions at these points are, without exception, parallel.

From the equations

$$z = \frac{x' - \lambda x}{1 - \lambda} = \frac{X' - \lambda X}{1 - \lambda}, \qquad \frac{dz}{dv} = \frac{\lambda'(X' - X)}{(1 - \lambda)^2},$$

we conclude that *the focal curve of the associated congruence is the envelope of the lines joining corresponding points* X', X *of the directrix curves of the surfaces.* It is also the locus of a point dividing the segment $X'X$ in the ratio $-\lambda(v)$

Case 2: $\lambda = const.$ Here we replace u by a new parameter $\bar{u}$ such that $\partial\bar{u}/\partial u = 1/\nu$. Equations (7) become

$$x'_u = \lambda x_u, \qquad x'_v = x_u + \lambda x_v,$$

and (16) reduces again to $x_{uu} = 0$. Hence we can write

$$x = \zeta'(v)u + \xi(v), \qquad x' = \lambda x + \zeta(v). \tag{31}$$

Equations (31), *where* λ *is an arbitrary constant,* $\neq 0$, *and* $\xi(v)$, $\zeta(v)$ *are arbitrary triples of functions, define the general parabolic map of two ruled surfaces whose generators correspond, in the case that the Combescurian ratio of the generators is constant.*

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* If λ were constant the surfaces would actually be homothetic.

The map is equivalent to a pseudo-homothetic transformation whose coefficient of stretching, λ, is constant, but whose parameters of translation vary from ruling to ruling. These parameters can be thought of as the point coordinates of the curve $\zeta = \zeta(v)$, whose tangents furnish the directions of the corresponding rulings of the two surfaces. It is to be noted that in this case there is no pair of curves along which the map has a singular behavior.

From the equations,

$$z = \frac{x' - \lambda x}{1 - \lambda} = \frac{\zeta}{1 - \lambda}, \qquad \frac{dz}{dv} = \frac{\zeta'}{1 - \lambda},$$

it follows that *the focal curve of the associated congruence is the envelope of lines which are parallel to corresponding generators and divide their common perpendiculars in the ratio* $-\lambda$. It is homothetic to the curve $\zeta = \zeta(v)$.

If $\lambda = -1$, the line-segments joining corresponding points of the two surfaces are all bisected by the focal curve. If $\lambda = 1$, the lines joining corresponding points of parallel rulings are all parallel; there is no focal curve. In these two cases and in these only the map is equiareal.

THEOREM 22. *A parabolic map is equiareal if and only if the surfaces are ruled surfaces whose rulings correspond and have the Combescurian ratio* ± 1.

Equations (30) and (31), by proper manipulation, can both be written:

$$(32) \quad x = [F'(v) - \lambda(v)\,\Phi'(v)]u + \Phi(v), \quad x' = \lambda(v)[F'(v) - \lambda(v)\Phi'(v)]u + F(v),$$

where $\lambda(v)$ is an arbitrary scalar function or constant, $\neq 0$, and $F(v)$, $\Phi(v)$ are arbitrary triples of functions for which $F' \not\equiv \lambda\Phi'$.

III. FUNDAMENTAL SYSTEMS OF CURVES FOR A NON-CONFORMAL MAP

14. **Corresponding orthogonal systems. Isometrically mapped systems.** Let the equations $x = x(u, v)$, $x' = x'(u, v)$, represent an arbitrary real map, not necessarily by parallel normals, of two arbitrary real surfaces. If the map is not conformal, there exist on the surfaces unique orthogonal systems of curves which correspond. If the map is not isometric, there exists on each surface a unique system such that the curves of the two systems correspond and are mapped isometrically. The two families of curves of each system are not necessarily real or distinct; for a conformal map, for example, they consist of the minimal curves.

THEOREM 23. *In a non-conformal map the corresponding orthogonal systems bisect the angles between the isometrically mapped systems, or, if the families constituting the latter are coincident, the corresponding orthogonal systems consist of these families and their orthogonal trajectories.*

For, if the families of isometrically mapped curves are distinct and parametric,

$$E' = E \neq 0, \quad F' \neq F, \quad G' = G \neq 0.$$

The differential equation of the corresponding orthogonal systems is, then, $Edu^2 - Gdv^2 = 0$.

If the families of isometrically mapped curves coincide, in the u-curves, and if on S their orthogonal trajectories are the v-curves, $E' = E$ and $F = 0$. But it follows, then, that $F' = 0$.

In the second case the map is never equiareal, since, by hypothesis, $G' \neq G$. In the first case the map is equiareal if and only if $F' = -F$, or $F'/\sqrt{E'G'} = -F/\sqrt{EG}$.

THEOREM 24. *In a non-isometric map the angles between the isometrically mapped curves at corresponding points are supplementary when and only when the map is equiareal.*

Since the map is not isometric, F and F' are not both zero. Hence the isometrically mapped curves never form orthogonal systems on both surfaces.

Theorem 24 gains in significance in light of the following fact.

THEOREM 25. *If a non-isometric map is equiareal the families of isometrically mapped curves are real and distinct.*

For, if the corresponding orthogonal systems of curves are parametric, $F' = F = 0$, $E'G' = EG$, and E, E', G, G' are all real and positive. Hence, for the differential equation of the isometrically mapped systems,

$$(E' - E)du^2 + (G' - G)dv^2 = 0,$$

it follows that $(E' - E)(G' - G) < 0$.

Parallel maps. If a parallel map is non-parabolic, the differential equations of the corresponding orthogonal systems and the isometrically mapped systems are, respectively,

$$\lambda EFdu^2 + (\lambda + \mu)EGdudv + \mu FGdv^2 = 0, \tag{33}$$

$$(\lambda^2 - 1)Edu^2 + 2(\lambda\mu - 1)Fdudv + (\mu^2 - 1)Gdv^2 = 0. \tag{34}$$

In discussing these systems it will be tacitly assumed that the map is non-conformal or non-isometric, according to the case in point.

THEOREM 26. *In a parallel map the corresponding orthogonal systems bisect the angles of the basic conjugate systems if and only if* $I = -1$.

This follows from (33), or from Theorem 8, in the non-parabolic case. In the parabolic case, the relationship, in the limiting form it would then have, is clearly impossible.

Since, when $I = -1$, the corresponding orthogonal systems bisect the angles of both the basic conjugate systems and the isometrically mapped systems, it is pertinent to inquire when the latter systems coincide.

THEOREM 27. *In a parallel map the isometrically mapped systems coincide with the corresponding conjugate systems when and only when the two surfaces are adjoint translation surfaces with real generators.**

For, equation (34) reduces to $dudv = 0$, if and only if $\lambda = \pm 1$, $\mu = \mp 1$. Moreover, if the isometrically mapped curves of a parabolic map coincided with the single families of basic parallel curves, the corresponding orthogonal systems would consist, by Theorem 23, of these families and their orthogonal trajectories; there would then be two families of corresponding parallel curves,—a contradiction.

The corresponding orthogonal systems coincide with the basic conjugate systems, if and only if the latter are the lines of curvature. In this case, if the lines of curvature are to be the only corresponding orthogonal systems, the invariant cannot be -1; cf. Theorem 17.

From Theorem 23 it is clear that the corresponding orthogonal systems never coincide with the isometrically mapped systems.

15. **Angles associated with the fundamental systems of curves.** Let θ be the angle between corresponding tangents to corresponding curves of the corresponding orthogonal systems of a non-conformal, non-parabolic map.† We find that

$$(35)\qquad \cos^2\theta = \frac{(I+1)^2H^2}{(I-1)^2EG + 4IH^2}, \qquad \tan^2\theta = \left(\frac{I-1}{I+1}\right)^2\frac{F^2}{H^2}.$$

Consequently, if the angle between the curves of the basic conjugate systems is denoted by φ,‡

$$(36)\qquad \tan^2\theta\tan^2\varphi = \left(\frac{I-1}{I+1}\right)^2.$$

If the corresponding orthogonal systems are the basic conjugate systems, $\theta = 0$ (or π) and $\varphi = \pi/2$, and conversely. If the curves of the corresponding orthogonal systems are cross-parallel, $\theta = \pi/2$ and $I = -1$, and conversely;

* Cf. Theorem 19. In the case of two adjoint translation surfaces with conjugate-imaginary generators, the generators are mapped pseudo-isometrically: the ratio of the squares of corresponding elements of arc is -1.

† If the map is inversely parallel there are two angles θ, supplementary to each other; which is chosen is immaterial for our purposes.

‡ In the elliptic case, F is positive and H^2, $\tan^2\varphi$ and $\left(\frac{I-1}{I+1}\right)^2$ are negative.

cf. Theorem 8. In all other cases the three expressions in (36) have finite values, not zero.

THEOREM 28. *If corresponding curves, C and C', of the corresponding orthogonal systems of a non-conformal, non-parabolic map are neither parallel nor perpendicular at corresponding points, the angle θ between C and C', the angle φ between the curves of the basic conjugate systems, and the invariant I are related by* (36).

Under the restrictions, $\theta \neq 0, \pi/2$, two of the quantities θ, φ, I determine the third; if two are constant, so is the third.

16. **The angles for an equiareal map.** *Non-parabolic case.* Assume, now, that the map is equiareal and let ψ be the angle between the isometrically mapped curves on either surface.

For an *inversely parallel* map,

$$\sin^2\psi = -\frac{4IH^2}{(I-1)^2EG}, \qquad \tan^2\psi = -\frac{4IH^2}{(I-1)^2EG+4IH^2}.$$

From these equations and (35) we find that

$$\frac{\tan^2\psi}{\cos^2\theta} = -\frac{4I}{(I+1)^2}, \qquad \frac{\sin^2\psi}{\sin^2\varphi} = -\frac{4I}{(I-1)^2}. \tag{37}$$

Elimination of ψ yields (36). Elimination of I gives the equation

$$\sin^2\theta = \frac{\cos^2\varphi}{\cos^2\psi}. \tag{38}$$

Of the various conclusions which can be drawn from (37) and (38), the following appear to be the most important. The map is assumed throughout to be inversely parallel, equiareal, and non-conformal.

THEOREM 29. *The square of the sine of the angle θ between corresponding curves of corresponding orthogonal systems equals the ratio of the squares of the cosines of the angles, φ and ψ, between the basic conjugate curves and the isometrically mapped curves, respectively.*

THEOREM 30. *The basic conjugate systems are the lines of curvature if and only if*

$$\tan^2\psi = -\frac{4I}{(I+1)^2}. \tag{39}$$

THEOREM 31. *If corresponding curves of the corresponding orthogonal systems are neither parallel nor perpendicular at corresponding points, any two of the four*

quantities, θ, φ, ψ, I, are determined when the other two are known and if any two are constant, so are the other two.

In the case of a *directly parallel* map,

$$\sin^2\psi = \frac{4IH^2}{(I-1)^2EG + 4IH^2}, \qquad \tan^2\psi = \frac{4IH^2}{(I-1)^2EG}.$$

Hence

$$\frac{\sin^2\psi}{\cos^2\theta} = \frac{4I}{(I+1)^2}, \qquad \frac{\tan^2\psi}{\sin^2\varphi} = \frac{4I}{(I-1)^2}. \tag{40}$$

As before, elimination of ψ gives (36). Elimination of I leads to

$$\sin^2\theta = \cos^2\varphi\cos^2\psi. \tag{41}$$

Thus, for a directly parallel non-parabolic map which is equiareal but not conformal, Theorem 31 is duplicated and Theorem 29 is valid if "ratio" is replaced by "product." Moreover, if the map is hyperbolic, Theorem 30 holds when the minus sign in (39) is replaced by a plus sign. If the map is elliptic, we find the following:

THEOREM 32. *Corresponding curves of corresponding orthogonal systems are cross-parallel, i. e., the surfaces are adjoint translation surfaces with imaginary generators,* if and only if*

$$\cos^2\varphi = \sec^2\psi. \tag{42}$$

Of equations (40), (41), the only one which involves angles which are real for both types of map is the first equation of (41); from it we easily show that the map is hyperbolic or elliptic, according as

$$\frac{\sin^2\psi}{\cos^2\theta} < 1 \quad \text{or} \quad \frac{\sin^2\psi}{\cos^2\theta} > 1. \tag{43}$$

Parabolic case. We are dealing here with the type of map considered in Theorem 22. One family of isometrically mapped curves on each surface coincides with the basic asymptotic lines,—the generators of the ruled surfaces. Consequently, since the corresponding orthogonal systems bisect the angles of the isometrically mapped systems, it follows that $\theta \pm \psi = \pi/2$.

THEOREM 33. *In a parabolic equiareal map the angle θ between corresponding curves of the corresponding orthogonal systems is the complement of, or differs by $\pi/2$ from, the angle ψ between the isometrically mapped curves.*

* Cf. Theorem 19 and the footnote to Theorem 27. The relation (42) takes the place, in this case, of the relation, $\cos^2\varphi = \cos^2\psi$, which holds in the case of adjoint translation surfaces with real generators.

Since, then, $\sin^2\psi = \cos^2\theta$, we obtain, in light of (43), the following:

THEOREM 34. *An equiareal, directly parallel map which is not conformal is hyperbolic, parabolic, or elliptic, according as* $\sin^2\psi/\cos^2\theta$ *is less than, equal to, or greater than unity.*

IV. CONTINUATION OF GENERAL THEORY

17. **Classification when parametric curves are arbitrary. Conditions of integrability.** If the surface $x = x(u, v)$, referred to two arbitrary real or conjugate-imaginary families of curves, is mapped by parallel normals on the surface $x' = x'(u, v)$, then

$$x'_u = \alpha x_u + \beta x_v, \qquad x'_v = \gamma x_u + \delta x_v, \tag{44}$$

where the pairs of functions $\alpha(u, v)$, $\delta(u, v)$ and $\beta(u, v)$, $\gamma(u, v)$ are real or conjugate-imaginary, according as u and v are real or conjugate-imaginary, and where

$$D = \alpha\delta - \beta\gamma \neq 0.$$

The basic conjugate systems are defined by the equation

$$\beta du^2 + (\delta - \alpha)du dv - \gamma dv^2 = 0, \tag{45}$$

whose discriminant is

$$\Delta = (\delta - \alpha)^2 + 4\beta\gamma = (\delta + \alpha)^2 - 4D.$$

Factoring (45) we obtain:*

$$2du + \left(\frac{\delta-\alpha}{\beta} + \sqrt{\frac{\Delta}{\beta^2}}\right) dv = 0, \quad 2du + \left(\frac{\delta-\alpha}{\beta} - \sqrt{\frac{\Delta}{\beta^2}}\right) dv = 0.$$

The cross ratio in which the directions defined by these equations, in the order given, are divided by the corresponding directions dx, dx', is

$$I = \frac{\dfrac{\alpha+\delta}{\beta} - \sqrt{\dfrac{\Delta}{\beta^2}}}{\dfrac{\alpha+\delta}{\beta} + \sqrt{\dfrac{\Delta}{\beta^2}}}. \tag{46}$$

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* The case $\beta = \gamma = 0$ is that already treated. We assume, then, that β and γ are not both zero and, in particular, that $\beta \neq 0$. The equations are written with each term of zero degree in α, β, γ, δ so that, when the surface S' is replaced by the surface $s = kx'$, k a constant, the invariant (46) will preserve its value, regardless of whether k is positive or negative.

We can now give, under our present hypothesis of arbitrary parametric curves, a complete classification of parallel maps. The map is directly or inversely parallel according as $D > 0$, or $D < 0$. It is hyperbolic, parabolic, or elliptic, according as $\Delta > 0$, $\Delta = 0$, or $\Delta < 0$.

In developing the conditions of integrability of (44), we choose as the parametric curves on S the asymptotic lines. Then, since (45) represents a conjugate system, $\alpha = \delta$, and our equations become

$$x'_u = \alpha\, x_u + \beta\, x_v, \qquad x'_v = \gamma\, x_u + \alpha\, x_v, \tag{47}$$

$$\beta du^2 - \gamma dv^2 = 0, \tag{48}$$

$$I = \frac{\dfrac{\alpha}{\beta} - \sqrt{\dfrac{\gamma}{\beta}}}{\dfrac{\alpha}{\beta} + \sqrt{\dfrac{\gamma}{\beta}}} = \frac{\dfrac{\alpha}{\gamma} - \sqrt{\dfrac{\beta}{\gamma}}}{\dfrac{\alpha}{\gamma} + \sqrt{\dfrac{\beta}{\gamma}}}, \tag{49}$$

where the different square roots in (49) are reciprocals of one another.*

Since
$$x'_{uv} = x'_{vu},$$

$$\gamma x_{uu} - \beta x_{vv} + (\gamma_u - \alpha_v) x_u + (\alpha_u - \beta_v) x_v = 0. \tag{50}$$

On substituting for x_{uu} and x_{vv} their values in terms of x_u and x_v, we obtain, as necessary and sufficient conditions for the integrability of (47), the equations:

$$\begin{aligned} \gamma_u - \alpha_v + \left\{ \begin{matrix} 11 \\ 1 \end{matrix} \right\} \gamma - \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\} \beta &= 0, \\ \beta_v - \alpha_u - \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\} \gamma + \left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\} \beta &= 0. \end{aligned} \tag{51}$$

18. **Existence of a map when one surface and I are given.** Given a surface S, referred to its asymptotic lines, and a point function $I(u, v)$, $\neq 1$, on the surface. Will there exist a surface S' on which S is mapped by parallel normals so that the invariant of the map is the given function I?

The problem is essentially that of solving the system of equations (49) and (51) for α, β, γ, when E, F, G, and I are given. Since we need, in particular, to determine the basic conjugate system (48), we seek an eliminant of (49) and (51) which involves, of the unknowns, only the ratio β/γ.

* The assumption, $\beta\gamma \neq 0$, of (49) rules out parabolic maps; but these have been considered in detail, in §5, in the typical case $\beta = 0$.

Case 1: $I = -1$. Here $\alpha = 0$ and equations (51) can be written as

$$(52)\qquad -\frac{\partial \log \gamma}{\partial u} = \begin{Bmatrix} 11 \\ 1 \end{Bmatrix} - \begin{Bmatrix} 22 \\ 1 \end{Bmatrix} \frac{\beta}{\gamma}, \quad \frac{\partial \log \beta}{\partial v} = \begin{Bmatrix} 11 \\ 2 \end{Bmatrix} \frac{\gamma}{\beta} - \begin{Bmatrix} 22 \\ 2 \end{Bmatrix}.$$

By differentiation, and application of the identity*

$$\frac{\partial}{\partial u} \begin{Bmatrix} 22 \\ 2 \end{Bmatrix} = \frac{\partial}{\partial v} \begin{Bmatrix} 11 \\ 1 \end{Bmatrix},$$

we obtain

$$(53)\qquad \frac{\partial^2 \log \beta/\gamma}{\partial u \partial v} = \frac{\partial}{\partial v}\left[\frac{\gamma}{\beta} \begin{Bmatrix} 11 \\ 2 \end{Bmatrix}\right] - \frac{\partial}{\partial v}\left[\frac{\beta}{\gamma} \begin{Bmatrix} 22 \\ 1 \end{Bmatrix}\right].$$

Each solution $\beta/\bar{\gamma}$ of this equation leads, by quadratures, to a surface S' associate to S in the sense of Bianchi and unique to within its homothetics. For, β and γ can be found from (52) and hence x' from $x'_u = \beta x_v, x'_v = \gamma x_u$.

Case 2: $I \neq -1$. In this case we rewrite (49) as

$$(54)\qquad \sqrt{\frac{\beta}{\gamma}} = \psi \frac{\alpha}{\gamma} \quad \text{or} \quad \sqrt{\frac{\gamma}{\beta}} = \psi \frac{\alpha}{\beta},$$

where

$$\psi = \frac{1-I}{1+I},$$

and (51) as

$$\frac{\beta_v}{\beta} + \begin{Bmatrix} 22 \\ 2 \end{Bmatrix} - \begin{Bmatrix} 11 \\ 2 \end{Bmatrix} \frac{\gamma}{\beta} = \frac{\alpha_u}{\beta}, \quad \frac{\gamma_u}{\gamma} + \begin{Bmatrix} 11 \\ 1 \end{Bmatrix} - \begin{Bmatrix} 22 \\ 1 \end{Bmatrix} \frac{\beta}{\gamma} = \frac{\alpha_v}{\gamma}.$$

From (54),

$$\psi \sqrt{\frac{\beta}{\gamma}} \frac{\alpha_u}{\beta} = \frac{\gamma_u}{\gamma} - \frac{1}{2} \frac{\partial}{\partial u} \log \psi^2 \frac{\gamma}{\beta}, \quad \psi \sqrt{\frac{\gamma}{\beta}} \frac{\alpha_v}{\gamma} = \frac{\beta_v}{\beta} - \frac{1}{2} \frac{\partial}{\partial v} \log \psi^2 \frac{\beta}{\gamma}.$$

Hence

$$-\psi \sqrt{\frac{\beta}{\gamma}} \frac{\beta_v}{\beta} + \frac{\gamma_u}{\gamma} = \frac{1}{2} \frac{\partial}{\partial u} \log \psi^2 \frac{\gamma}{\beta} + \psi \sqrt{\frac{\beta}{\gamma}}\left[\begin{Bmatrix} 22 \\ 2 \end{Bmatrix} - \begin{Bmatrix} 11 \\ 2 \end{Bmatrix} \frac{\gamma}{\beta}\right],$$

$$\frac{\beta_v}{\beta} - \psi \sqrt{\frac{\gamma}{\beta}} \frac{\gamma_u}{\gamma} = \frac{1}{2} \frac{\partial}{\partial v} \log \psi^2 \frac{\beta}{\gamma} + \psi \sqrt{\frac{\gamma}{\beta}}\left[\begin{Bmatrix} 11 \\ 1 \end{Bmatrix} - \begin{Bmatrix} 22 \\ 1 \end{Bmatrix} \frac{\beta}{\gamma}\right].$$

* Cf., e. g., Eisenhart, *Differential Geometry*, p. 189.

Solving for β_v/β and γ_u/γ, we obtain

$$(55) \qquad (1-\psi^2)\frac{\partial \log \beta}{\partial v} = A, \qquad (1-\psi^2)\frac{\partial \log \gamma}{\partial u} = B,$$

where

$$A = \frac{1}{2}\frac{\partial \log \psi^2 \frac{\beta}{\gamma}}{\partial v} + \frac{\partial}{\partial u}\left(\psi\sqrt{\frac{\gamma}{\beta}}\right) + \psi\sqrt{\frac{\gamma}{\beta}}\left[\begin{Bmatrix} 11 \\ 1 \end{Bmatrix} - \begin{Bmatrix} 22 \\ 1 \end{Bmatrix}\frac{\beta}{\gamma}\right]$$
$$+ \psi^2\left[\begin{Bmatrix} 22 \\ 2 \end{Bmatrix} - \begin{Bmatrix} 11 \\ 2 \end{Bmatrix}\frac{\gamma}{\beta}\right],$$

$$B = \frac{1}{2}\frac{\partial \log \psi^2 \frac{\gamma}{\beta}}{\partial u} + \frac{\partial}{\partial v}\left(\psi\sqrt{\frac{\beta}{\gamma}}\right) + \psi^2\left[\begin{Bmatrix} 11 \\ 1 \end{Bmatrix} - \begin{Bmatrix} 22 \\ 1 \end{Bmatrix}\frac{\beta}{\gamma}\right]$$
$$+ \psi\sqrt{\frac{\beta}{\gamma}}\left[\begin{Bmatrix} 22 \\ 2 \end{Bmatrix} - \begin{Bmatrix} 11 \\ 2 \end{Bmatrix}\frac{\gamma}{\beta}\right].$$

From (55) we have, finally, the equation

$$(56) \qquad \frac{\partial^2 \log \beta/\gamma}{\partial u \partial v} = \frac{\partial}{\partial u}\frac{A}{1-\psi^2} - \frac{\partial}{\partial v}\frac{B}{1-\psi^2},$$

which involves β and γ only in their ratio.

In discussing these results, we note first the effect of changing the determination of the square root, when an actual map is given. The order of the basic conjugate directions is reversed, I is replaced by $1/I$ and ψ changes sign. Thus (55) and (56) are unchanged.

If, for a fixed determination of the square root, β/γ is a particular solution of (56), equations (55) are integrable and determine β and γ except for the same multiplicative constant k; then α, unique except for the factor k, is obtained from (54). For these values of α, β, γ, equations (47) are integrable and determine x' except for the multiplier k and an additive triple.

Thus, in this case also, the determination of all the surfaces S' in question requires the solution of a partial differential equation of the second order and quadratures.

Theorem 35. *Given a surface S and a point function I, $\neq 1$, on it, there exist infinitely many surfaces S' which correspond to S by a parallel map whose invariant is I. If S is referred to its asymptotic lines and a particular solution, β/γ, of* (56)*—or of* (53), *if $I = -1$ —is chosen, S' is determined to within its homothetics and its point coördinates can be found by quadratures.*

A solution, β/γ, of (53) or (56) determines immediately, not only the basic conjugate systems (48), but also the asymptotic lines, $\beta du^2 + \gamma dv^2 = 0$, of the

surface S'. It is possible to replace β/γ by a quantity of even more striking geometrical significance, namely by the ratio of the radii R_{c_1}, R_{c_2} of normal curvatures of the surface S in the conjugate directions,

$$\sqrt{\frac{\beta}{\gamma}}\,du + dv = 0, \quad \sqrt{\frac{\beta}{\gamma}}\,du - dv = 0,$$

which serve, in the order given, as the basic directions. We find that

$$R = \frac{R_{c_1}}{R_{c_2}} = -\frac{E - 2F\sqrt{\dfrac{\beta}{\gamma}} + G\dfrac{\beta}{\gamma}}{E + 2F\sqrt{\dfrac{\beta}{\gamma}} + G\dfrac{\beta}{\gamma}}, \tag{57}$$

whence

$$\sqrt{\frac{\beta}{\gamma}} = \frac{(1-R)F \pm \sqrt{(1-R)^2F^2 - (1+R)^2EG}}{(1+R)G}. \tag{58}$$

For a given non-parabolic map the ratio R in question is determined by (57); hence the value of $\sqrt{\beta/\gamma}$ is given by (58) for a proper choice of sign. Substituting this value in (53) or (56), according as I is or is not -1, we get a valid equation in E, F, G, I (or ψ), and R. Conversely, if this equation is satisfied for one choice of sign, then $\sqrt{\beta/\gamma}$, given by (58) for this choice of sign, satisfies (53) or (56).

THEOREM 36. *A necessary and sufficient condition that there exist a surface S' on which a given non-minimal* surface S is mapped by parallel normals, so that the invariant is a prescribed function I, $\neq 1$, and the ratio of the radii of normal curvatures in the ordered basic conjugate directions is a prescribed function R, is that, when S is referred to its asymptotic lines, E, F, G, I, and R satisfy* (56)—*or, when $I = -1$,* (53)—*for one choice of sign of the radical in* (58).

There are certain preliminary restrictions which can be placed on $I(u, v)$ and $R(u, v)$. If S is of positive curvature, I must be real, whereas, if S is of negative curvature, I may be real or of the form $e^{\varphi(u,v)i}$. If $I \neq -1$, R is of the same type as I, whereas, if $I = -1$, R may be of either type. In case R is real, it must satisfy the conditions†

$$\frac{r_2}{r_1} \leqq R \leqq \frac{r_1}{r_2}, \qquad R \neq -1,$$

where r_1 and r_2 are the principal radii of curvature of S.

* If S were minimal, R would be always -1 and hence not dependent on β/γ.

† The restriction, $R \neq -1$, serves to prevent the basic conjugate directions from coinciding in an asymptotic direction.

19. **Existence of a non-parabolic map when the spherical representation is given.** It is well known that a system of curves $\mathfrak{C}$ on the Gauss sphere represents a conjugate system of curves C on each of infinitely many surfaces S. It can be shown further* that, if the curves $\mathfrak{C}$ are parametric,† each solution $\theta = M$ of the equation

$$(59)\qquad \frac{\partial^2 \log \theta}{\partial u \partial v} = \frac{\partial}{\partial u}\left[\begin{Bmatrix}12\\1\end{Bmatrix}_1 - \begin{Bmatrix}11\\2\end{Bmatrix}_1 \frac{1}{\theta}\right] - \frac{\partial}{\partial v}\left[\begin{Bmatrix}12\\2\end{Bmatrix}_1 - \begin{Bmatrix}22\\1\end{Bmatrix}_1 \theta\right],$$

where the Christoffel symbols pertain to the sphere, determines one of the required surfaces S to within its homothetics, in that it leads to values for e and g, which satisfy the equation

$$(60)\qquad \frac{e}{g} = M,$$

and are unique except for the same multiplicative constant. The ratio R_{c_1}/R_{c_2} of the radii of normal curvature of the surface in the directions of the conjugate system C is‡

$$(61)\qquad \frac{R_{c_1}}{R_{c_2}} = \frac{\mathfrak{G}}{\mathfrak{E}} M,$$

the asymptotic lines on the surfaces are defined by $M du^2 + dv^2 = 0$, and its point coördinates can be found by quadratures when those of the sphere are known.

From these facts we obtain immediately the following result.

THEOREM 37. *Two surfaces determined, to within their homothetics, by two distinct solutions M and M' of* (59) *correspond by a non-parabolic parallel map whose basic conjugate systems are represented by the given curves on the sphere and whose invariant I has the value M'/M.*

The spherical representation of the basic conjugate systems of a non-parabolic map can, then, be prescribed at pleasure. However, it is impossible, in general, to prescribe at the same time the value of I; the conditions under which there will exist a map in this case are evident from Theorem 37.

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* Author, *Spherical representation of conjugate systems and asymptotic lines*, to appear in the Annals of Mathematics.

† Explicit mention of this condition, which we shall always assume fulfilled, will henceforth be suppressed.

‡ $\mathfrak{E}$, $\mathfrak{F}$, $\mathfrak{G}$ are the differential coefficients of the first order of the sphere and $\mathfrak{H}^2 = \mathfrak{E}\mathfrak{G} - \mathfrak{F}^2$.

THEOREM 38. *A necessary and sufficient condition that a system of curves $\mathfrak{C}$ on the sphere represent the basic conjugate systems of two surfaces corresponding by a parallel map with prescribed invariant I is that there exist a function M such that M and IM both satisfy* (59). *The two surfaces are then determined to within their homothetics and their point coördinates can be found by quadratures, when those of the sphere are known.*

If for θ in (59) e/g and Ie/g are substituted in turn, the difference of the two equations obtained reduces, by virtue of the identities*

$$(62) \qquad -\frac{g}{e}\left\{\begin{matrix}11\\2\end{matrix}\right\}_1 = \left\{\begin{matrix}12\\1\end{matrix}\right\}, \qquad -\frac{e}{g}\left\{\begin{matrix}22\\1\end{matrix}\right\}_1 = \left\{\begin{matrix}12\\2\end{matrix}\right\},$$

to the fundamental equation (11) for a non-parabolic map.

If h and k are the point invariants of the conjugate system, then, by (62),

$$(63) \qquad h-k = -\frac{\partial}{\partial u}\left[\frac{g}{e}\left\{\begin{matrix}11\\2\end{matrix}\right\}_1\right] + \frac{\partial}{\partial v}\left[\frac{e}{g}\left\{\begin{matrix}22\\1\end{matrix}\right\}_1\right].$$

COROLLARY 1. *The given system $\mathfrak{C}$ gives rise to a map of invariant I for which the basic conjugate systems both have equal point invariants if and only if M exists such that M and IM both satisfy the equations*

$$(64) \qquad \frac{\partial^2 \log\theta}{\partial u\partial v} = \frac{\partial}{\partial v}\left\{\begin{matrix}12\\1\end{matrix}\right\}_1 - \frac{\partial}{\partial u}\left\{\begin{matrix}12\\2\end{matrix}\right\}_1, \qquad \frac{\partial}{\partial u}\left[\frac{1}{\theta}\left\{\begin{matrix}11\\2\end{matrix}\right\}_1\right] = \frac{\partial}{\partial v}\left[\theta\left\{\begin{matrix}22\\1\end{matrix}\right\}_1\right].$$

If, in particular, the surfaces are to be translation surfaces, the second of these equations must be replaced by the restrictions, $\left\{\begin{matrix}11\\2\end{matrix}\right\}_1 = 0$, $\left\{\begin{matrix}22\\1\end{matrix}\right\}_1 = 0$, on the curves $\mathfrak{C}$. Then each solution $\theta = M$ of the first equation determines a map, provided merely that the given function I is of the form $U(u)/V(v)$; cf. §9.

If the surfaces are to be associate surfaces of Bianchi, it suffices that (64) be compatible; for if $\theta = M$ is a common solution, so also is $\theta = IM = -M$.

* Cf., e. g., Eisenhart, *Differential Geometry*, p. 201.

Parallel maps whose basic systems are isothermal-conjugate* can be treated, in light of (60), by means of Theorem 38.

COROLLARY 2. *The given system $\mathfrak{C}$ gives rise to a map of invariant I for which the basic systems are isothermal-conjugate if, and only if, I is of the form $U(u)/V(v)$ and there exists a function M of this form such that I and IM both satisfy the equation*

$$\frac{\partial}{\partial u}\left[\left\{\begin{matrix}12\\1\end{matrix}\right\}_1 - \left\{\begin{matrix}11\\2\end{matrix}\right\}_1\frac{1}{\theta}\right] = \frac{\partial}{\partial v}\left[\left\{\begin{matrix}12\\2\end{matrix}\right\}_1 - \left\{\begin{matrix}22\\1\end{matrix}\right\}_1\theta\right].$$

Substituting e/g for θ in (59) and applying (63), we obtain the relation

$$\frac{\partial^2 \log e/g}{\partial u\partial v} = (h-k) + (h_1-k_1),$$

where h_1, k_1 are the plane invariants of the conjugate system. Hence:

THEOREM 39. *The basic conjugate systems of a non-parabolic map are both isothermal-conjugate if, and only if, the differences of their point invariants are equal to each other and to the difference of their common plane invariants taken in opposite order.*

Lines of curvature basic. If the given system of curves $\mathfrak{C}$ on the sphere is orthogonal, (59) can be replaced by†

$$(65)\qquad 2\,\frac{\partial^2 \log\theta}{\partial u\partial v} = \frac{\partial}{\partial u}\left[\left(\frac{1}{\theta}-1\right)\frac{\partial \log \mathfrak{E}}{\partial v}\right] - \frac{\partial}{\partial v}\left[(\theta-1)\,\frac{\partial \log \mathfrak{G}}{\partial u}\right].$$

Each solution $\theta = R$ of this equation determines, to within its homothetics, a surface whose lines of curvature are represented by the curves $\mathfrak{C}$. Moreover, the ratio r_1/r_2 of the principal radii of curvature of the surface is precisely R:

$$(66)\qquad \frac{r_1}{r_2} = R.$$

THEOREM 40. *Two surfaces determined to within their homothetics by two distinct solutions R and R′ of (65) are mapped by parallel normals so that the lines of*

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* A conjugate system is termed isothermal-conjugate if, and only if, when it is parametric, e/g is of the form $U(u)/V(v)$.

† Author, loc. cit.

curvature correspond and are represented by the given orthogonal system on the sphere; the invariant I of the map has the value R'/R.

The analogue of Theorem 38, stated in brief, is as follows:

THEOREM 41. *A necessary and sufficient condition that there exist two surfaces corresponding by a parallel map with prescribed invariant I so that the lines of curvature are basic and are represented by a given orthogonal system $\mathfrak{E}$ on the sphere is that a function R exist such that R and IR both satisfy* (65).

From the identities

$$\frac{\partial \log E}{\partial v} = \frac{r_2}{r_1}\frac{\partial \log \mathfrak{E}}{\partial v}, \qquad \frac{\partial \log G}{\partial u} = \frac{r_1}{r_2}\frac{\partial \log \mathfrak{G}}{\partial u}, \tag{67}$$

follows the

COROLLARY. *The given orthogonal system $\mathfrak{E}$ gives rise to a map, of invariant I, of two isometric surfaces if, and only if, I is of the form $U(u)/V(v)$ and a function R exists such that $R^2\mathfrak{E}/\mathfrak{G}$ is of this form and R and IR satisfy the equation*

$$\frac{\partial}{\partial u}\left(\frac{1}{\theta}\,\frac{\partial \log \mathfrak{E}}{\partial v}\right) = \frac{\partial}{\partial v}\left(\theta\,\frac{\partial \log \mathfrak{G}}{\partial u}\right).$$

The equation,

$$\frac{\partial^2 \log r_1/r_2}{\partial u \partial v} = (h - k) - (h_1 - k_1),$$

which is obtained by substituting r_1/r_2 for θ in (65) and applying (67), gives the following interesting result:

THEOREM 42. *The ratios of the principal curvatures of two surfaces which are mapped by parallel normals so that the lines of curvature correspond (and are parametric) are both of the form $U(u)/V(v)$ if, and only if, the differences of the point invariants of the lines of curvature are equal to each other and to the difference of their common plane invariants.**

20. **Existence of a parabolic map when the spherical representation is given.** The u-curves of a parametric system of real curves on the sphere represent one family of asymptotic lines on each of infinitely many surfaces of negative curvature. In particular,† each solution $\theta = L$ of the equation

$$\frac{\partial^2\theta}{\partial u^2} - \frac{\partial}{\partial u}\left[\left\{ {12 \atop 1} \right\}_1 - \left(\left\{ {11 \atop 1} \right\}_1 - 2\left\{ {12 \atop 2} \right\}_1\right)\theta - 2\left\{ {11 \atop 2} \right\}_1 \theta^2\right] + \frac{\partial}{\partial v}\left[\left\{ {12 \atop 2} \right\}_1 - \left\{ {11 \atop 2} \right\}_1 \theta\right] = 0 \tag{68}$$

* The results of this section can be amplified by further applications of the theory set forth in the paper cited.

† Author, loc. cit.

determines one of the surfaces S to within its homothetics, in that it leads to values of f and g which satisfy the equation

$$\frac{g}{2f} = L \tag{69}$$

and are unique except for the same constant multiplier. The point coördinates of the surface can be found by quadratures when those of the sphere are known. Finally, the second family of asymptotic lines is defined by $du + Ldv = 0$, and the cotangent of the angle φ between the asymptotic lines is

$$\cot \varphi = \frac{\mathfrak{E}L - \mathfrak{F}}{\mathfrak{H}}. \tag{70}$$

THEOREM 43. *Two surfaces determined by two distinct solutions L and L' of* (68) *correspond by a parabolic map whose basic asymptotic lines are represented by the u-curves of the given system on the sphere and whose invariant J has the value*

$$J = 2(L' - L)\frac{\mathfrak{E}}{\mathfrak{H}}.$$

The value of J is found from (70) by applying Theorem 4. It is clear that we can choose arbitrarily the spherical representation of a parabolic map, specifying the family of curves which is to correspond to the basic asymptotic lines. We cannot at the same time choose J at pleasure, as the following theorem shows.

THEOREM 44. *A necessary and sufficient condition that the u-curves of a parametric system of real curves $\mathfrak{C}$ on the sphere represent the basic asymptotic lines on two surfaces corresponding by a parabolic map which has the given spherical representation and a prescribed invariant J is that there exist a function L such that L and $L + 2J\mathfrak{H}/\mathfrak{E}$ satisfy* (68).

To apply the theorem to ruled surfaces, we note that, for a surface for which $e = 0$,*

$$\begin{Bmatrix} 11 \\ 2 \end{Bmatrix} + \begin{Bmatrix} 11 \\ 2 \end{Bmatrix}_1 = 0, \tag{71}$$

and that the surface is ruled if, and only if, one and hence both of these symbols vanish.

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* Cf., e. g., Eisenhart, *Differential Geometry*, p. 162.

COROLLARY. *The given system $\mathfrak{C}$ leads in the manner prescribed to a parabolic map, of invariant J, of two ruled surfaces whose rulings correspond if, and only if,* $\left\{ {11 \atop 2} \right\}_1 = 0$ *and L exists such that L and $L + 2J\mathfrak{H}/\mathfrak{C}$ both satisfy the equation*

$$\frac{\partial^2\theta}{\partial u^2} - \frac{\partial}{\partial u}\left[\left\{ {12 \atop 1} \right\}_1 - \left(\left\{ {11 \atop 1} \right\}_1 - 2\left\{ {12 \atop 2} \right\}_1\right)\theta\right] + \frac{\partial}{\partial v}\left\{ {12 \atop 2} \right\}_1 = 0.$$

It is to be noted that this equation, unlike (68), is linear in θ.

If for θ in (68) $g/2f$ and $g/2f + 2J\mathfrak{H}/\mathfrak{C}$ are substituted in turn, the difference of the two resulting equations can be reduced, by the use of (71) and analogous identities, to the fundamental equation (19) of a parabolic map.

21. **Miscellaneous theorems. Generalizations.** The product of two parallel maps, $S \longrightarrow S'$ and $S' \longrightarrow S''$, is a parallel map, $S \longrightarrow S''$. If the given maps are non-parabolic and the same conjugate system on the surface S' is basic for both, the product is non-parabolic; its basic conjugate systems, on the surfaces S and S'', are those of the given maps, and its invariant is the product of the invariants of the given maps. If the given maps are parabolic and the same family of asymptotic lines on S' is basic for both, the product is parabolic.

If a surface $x = x(u, v)$ is mapped by parallel normals on a surface $y = y(u, v)$ and also on a surface $z = z(u, v)$, it is mapped by parallel normals on every surface $my + nz$, where m and n are arbitrary constants, not both zero. If the given maps are non-parabolic and the same conjugate system on the surface x is basic for both, the map $x \longrightarrow my + nz$ is non-parabolic, and the conjugate system in question is basic for it; moreover, if the Combescurian ratios for the maps $x \longrightarrow y$ and $x \longrightarrow z$ are, respectively, λ, μ and λ', μ', those for the map $x \longrightarrow my + nz$ are $m\lambda + n\lambda'$, $m\mu + n\mu'$. Similar observations can be made in case the given maps are parabolic and the same family of asymptotic lines on the surface x is basic for both.

Of particular interest is the following theorem, which we state without proof.

THEOREM 45. *If the maps $x \longrightarrow y$ and $x \longrightarrow z$ have the same invariant I, $\neq -1$, and just one map $x \longrightarrow my + nz$, $mn \neq 0$, has this invariant, the surfaces y and z are homothetic. However, if $I = -1$, the invariant of every map $x \longrightarrow my + nz$ is -1.**

The elements on which the general theory of parallel maps, given in §§2–4, 17, 18, is based, namely, conjugate systems, asymptotic lines, and the properties of one-dimensional linear correspondences, are invariant under projective

* This latter fact has been pointed out by Eisenhart, p. 508 of paper cited in §1.

transformations and polar reciprocations. Consequently, the theory can be generalized by transformations of these two types.

In the first case the transformed map consists of two surfaces whose tangent planes at corresponding points, P and P', intersect in a line l of a fixed plane. The basis of the classification is the projective correspondence of the points of l established by the pairs of corresponding tangents at P and P' to the surfaces. The associated congruence, as in the case of a parallel map, is made up of the lines PP'.

In the second case a line l joining corresponding points, P and P', of the transformed surfaces passes through a fixed point. The basis of the classification is the projective correspondence of the planes through l established by the pairs of corresponding tangents at P and P' to the surfaces. The associated congruence consists of the lines of intersection of the tangent planes at corresponding points P and P'.

In both cases the essential content of the existence theorems of §§4, 18 is valid for the transformed map. In particular, we recall (§4) that the fundamental equation (14) of a non-parabolic map is invariant under a projective transformation.

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